Morse index properties of colliding solutions to the N-body problem

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Abstract

We study a singular Hamiltonian system with an α-homogeneous potential that contains, as a particular case, the classical N–body problem. We introduce a variational Morse–like index for a class of collision solutions and, using the asymptotic estimates near collisions, we prove the non-minimality of some special classes of colliding trajectories under suitable spectral conditions provided α is sufficiently away from zero. We then prove some minimality results for small values of the parameter α.

1 Introduction

In this paper we consider the second order Hamiltonian system

$$M\ddot{x} = \nabla U(x) \tag{1}$$

with

$$U(x) := \sum_{\substack{i,j=1 \\ i<j}}^{N} U_{ij}(x_i - x_j), \quad U_{ij}(x_i - x_j) = \frac{m_i m_j}{|x_i - x_j|^\alpha}, \tag{2}$$

$x = (x_1, \ldots, x_N) \colon [0,1] \to \mathbb{R}^{Nd}$, $d \geq 2$, $N \geq 2$, $\alpha \in (0,2)$, and $M = \mathrm{diag}[m_1, \ldots, m_N]$. This system describes the well-known generalized N–body problem, namely the motion of N particles $x_1, \ldots x_N$ of positive masses $m_1, \ldots, m_N$ under the external force ∇U due to the generalized Kepler potential (2). The classical Keplerian case corresponds to the value $\alpha = 1$. It is a classical result, see [16, 27], that if x is a solutions of (1) on $[0,1)$ and if x cannot be extended to the whole interval $[0,1]$, then $\lim_{t \to 1} U(x(t)) = +\infty$; moreover, if $\|x\|$ remains bounded, then there must be a collision at $t = 1$, i.e. there exist two different indices i, j with $|x_i(t) - x_j(t)| \to 0$ as $t \to 1$.

It is evident that (1) has a rather delicate variational structure, since the Euler–Lagrange action functional

$$\mathcal{A}_N(x) = \frac{1}{2} \int_0^1 |\dot{x}(t)|^2 \, dt + \int_0^1 U(x(t)) \, dt \tag{3}$$

may blow up along orbits $x(\cdot)$ that approach the collision set

$$\Delta = \{x \in \mathbb{R}^{Nd} : \exists i \neq j, \ x_i = x_j\}. \tag{4}$$

Several recent papers are concerned with existence and qualitative properties of *collisionless solu-tions*, i.e. solutions such that $x(t) \notin \Delta$ for all t. A first approach to avoid collision solutions is the introduction of the so-called *strong force* assumption $\alpha \geq 2$ (see [12]). This constraint makes it possible to prove that the Palais–Smale condition holds and to find non-collision solutions by some standard tool of Critical Point Theory. Unfortunately, the Keplerian case does not satisfies such a condition, for this reason much attention has been paid to the complementary case $\alpha \in (0, 2)$. The bibliography about this problem is huge, concerning the variational approach we cite, among others, [2, 3, 7, 8, 9, 11, 20, 21, 25].

In this paper we deal with some variational properties of solutions to (1) possessing an isolated collision. Roughly speaking, we will give an estimate of a generalized Morse index by means of the asymptotic behavior of such a solution near the collision. It is known that the action functional lacks regularity at collision orbits, so that the usual Morse index cannot be defined. This problem was overcome in [9] by the technique of approximate solutions. One of the results of that paper is an upper bound on the number of total collisions for periodic solutions that can be suitably approximated in the H^1–sense by solutions corresponding to a regularized potential. The proof relies on the construction of suitable variations introduced in [20]. Later, Riahi (see [18]) generalized this result to solutions with partial collisions, essentially by using the same method. We also cite the paper [24], where the author proves the existence of one classical periodic solution in the case $1 < \alpha < 2$ and of one generalized periodic solution with at most one collision in the case $0 < \alpha \leq 1$. The existence is proved again by the method of approximate solutions, and the Author supplies some estimates on the Morse index of these approximations. In the quoted papers, one of the main ideas is that whenever the ratio between the dimension of the space and the number of bodies involved in the collision is big enough, then a collision gives a contribution to the Morse index of the corresponding trajectory.

The main novelty of this paper consists in the use of the asymptotic behavior near a collision in order to give an estimate on the Morse index. After fixing notation and reviewing some known facts (Section 2), we introduce in Section 3 the variational setting of our problem and recall the main asymptotic estimates (see [22, 11, 26]) that will be used to prove our main results. In the next section we define the generalized Morse index and provide in Theorem 4.3 a sufficient spectral condition on the asymptotic configuration, ensuring that orbits with a single collision have an infinite index. Theorem 4.3 should be compared to the results of [10], where non–minimality for a different class of colliding solutions is proved. In particular, no condition like (28) appears in that paper where, imposing reasonable assumptions about central configurations and a symmetry assumption on the perturbation (the Author perturbs the N-body potential with a term which is strongly dominated in a C^2 sense near the collision set by the Newtonian potential), it is shown that periodic orbits can be found with the calculus of variations approach which avoid binary or triple collisions. An additional assumption avoids total collapse orbits.

Section 5 is devoted to showing that the condition of the last section is satisfied in some important cases, e.g. the collinear central configuration of three equal masses or the polygonal configurations for N masses. In particular, we will show that our abstract theorem applies for every α lying outside a small neighborhood of $\alpha = 0$. Finally, Section 6 is somewhat complementary to the previous ones. Indeed, we analyze in deeper detail what happens in the limit $\alpha \to 0$, and prove that under suitable assumptions, families $(x_\alpha)_\alpha$ of one-collision solutions are "minimal", in the sense that the second derivative of the action along compactly supported variations is positive.

2 Preliminaries

We consider the generalized keplerian potential defined in (2) on *collisionless configurations* $x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd} \setminus \Delta$, where Δ is the collision set defined in (4). The classical keplerian inter-

action corresponds to the choice $\alpha = 1$. We study the dynamical system (1), recalling that it is conservative system, in the sense that the *total energy*

$$h = \frac{1}{2}\langle M\dot{x}, \dot{x}\rangle - U(x) = \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{x}_i|^2 - U(x) \tag{5}$$

is constant along solutions. Since the *center of mass* moves with a uniform motion, without loss of generality, we can fix it at the origin, that is

$$\sum_{i=1}^{N} m_i x_i = 0.$$

The potential (2) will be then defined on the configuration space

$$\Lambda = \left\{ x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd} \setminus \Delta : \sum_{i=1}^{N} m_i x_i = 0 \right\} \tag{6}$$

For any $x \in \mathbb{R}^{Nd}$, the *moment of inertia* is defined by

$$I = I(x) = \langle Mx, x\rangle = \sum_{i=1}^{N} m_i|x_i|^2$$

and its gradient is simply
$$\nabla I(x) = 2Mx.$$

The following definition is quite standard.

Definition 2.1. *A* central configuration *is a critical point of the function U constrained to the set $\mathcal{E} = \{x \in \Lambda \mid I(x) = 1\}$. We will call $\mathcal{E}$ the* standard ellipsoid.

Remark 2.2. *An equivalent definition of central configuration is the following: $\bar{x} \in \mathcal{E}$ is a central configuration if there exists a solution of (1) of the form $x(t) = \phi(t)\bar{x}$, for some real-valued C^2-function ϕ. For these classical facts, we refer to [17, 26].*

Let us denote the radial and the angular components of $x \in \mathbb{R}^{Nd} \setminus \{0\}$ by

$$r(x) = \sqrt{I(x)}, \qquad s(x) = \frac{x}{\sqrt{I(x)}}.$$

In particular $\mathcal{E}$ is now described by the simple condition $r = 1$. Since $U_{|\mathcal{E}}(s) = r^{\alpha}U(rs) = I^{\frac{\alpha}{2}}(x)U(x)$, it follows easily that

$$\nabla U_{|\mathcal{E}}(s) \cdot v = \frac{\alpha}{2}I^{\frac{\alpha}{2}-1}(x)U(x)\nabla I(x) \cdot v + I^{\frac{\alpha}{2}}(x)\nabla U(x) \cdot v, \tag{7}$$

and

$$\nabla^2 U_{|\mathcal{E}}(s)(v, v) = I^{\frac{\alpha}{2}}(x)\nabla^2 U(x)(v, v) + \alpha I^{\frac{\alpha}{2}-1}(x)\langle\nabla I(x), v\rangle\langle\nabla U(x), v\rangle$$
$$+ \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right)I^{\frac{\alpha}{2}-2}(x)U(x)\langle\nabla I(x), v\rangle^2 + \frac{\alpha}{2}I^{\frac{\alpha}{2}-1}(x)U(x)\nabla^2 I(x)(v, v). \tag{8}$$

As a consequence, when x is a central configuration, that is when $x \in \mathcal{E}$ and $\nabla U_{|\mathcal{E}}(x) \cdot v = 0$ for every $v \in T_x\mathcal{E}$, we deduce from (7) that $\nabla U(x) = -\alpha U(x)Mx$. Replacing in (8) we get

$$\nabla^2 U_{|\mathcal{E}}(s)(v, v) = \nabla^2 U(x)(v, v) - \alpha(\alpha + 2)U(x)\langle Mx, v\rangle^2 + \alpha U(x)\langle Mv, v\rangle.$$

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Since $v \in T_s\mathcal{E}$, we must have $\langle v, Mx \rangle = 0$, therefore the expression for the second derivative of $U_{|\mathcal{E}}$ evaluated at s is, for any $v \in T_s\mathcal{E}$ with $\sum_i m_i v_i = 0$,

$$\nabla^2 U_{|\mathcal{E}}(s)(v, v) = \nabla^2 U(x)(v, v) + \alpha U(x)\langle Mv, v \rangle, \tag{9}$$

where

$$\nabla^2 U(x)(v, v) = \alpha \sum_{i < j} m_i m_j \left[(\alpha + 2) \frac{\langle x_i - x_j, v_i - v_j \rangle^2}{|x_i - x_j|^{\alpha + 4}} - \frac{|v_i - v_j|^2}{|x_i - x_j|^{\alpha + 2}} \right] \tag{10}$$

is the Hessian of U on the whole space $\mathbb{R}^{Nd} \setminus \Delta$. When each v_j is orthogonal to the vector space generated by $\{x_1, \dots, x_N\}$, we deduce from (10) that the Hessian of the potential U is simply

$$\nabla^2 U(x)(v, v) = -\alpha \sum_{i < j} m_i m_j \frac{|v_i - v_j|^2}{|x_i - x_j|^{\alpha + 2}} = -\alpha \langle v, Av \rangle, \tag{11}$$

where

$$A(x) = [a_{ij}(x)], \quad a_{ij}(x) = \begin{cases} \displaystyle \sum_{k \neq i} \frac{m_k}{|x_i - x_k|^{\alpha + 2}} & i = j \\ \displaystyle -\frac{m_j}{|x_i - x_j|^{\alpha + 2}} & i \neq j. \end{cases} \tag{12}$$

Remark 2.3. *Every tangent vector $v \in T_s\mathcal{E}$ can be seen as an N-uple of vectors $(v_1, \dots, v_N)$, where each v_j stands for the position of the j-th particle in the euclidean space $\mathbb{R}^d$. This justifies the slight abuse of looking at the Hessian $\nabla^2 U_{|\mathcal{E}}(s)$ as a quadratic form on $\mathbb{R}^N$.*

In this case the expression for the constrained second derivative (9) can be recast as

$$\nabla^2 U_{|\mathcal{E}}(s)(v, v) = \alpha \left(-\langle v, MAv \rangle + U(x)\langle Mv, v \rangle \right) \tag{13}$$

for all $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ with $\sum_{i=1}^N m_i v_i = 0$,

3   The variational setting

It is well known that standard Critical Point Theory cannot be applied to find solutions of (1) possessing a collision. Indeed, the presence of collisions along a trajectory makes the action function $\mathcal{A}_N$ (see definition (3) below) possibly meaningless. As such, it might even be impossible to say that a collision solution is a critical point of $\mathcal{A}_N$. For this reason, let us define the function spaces

$$\Omega = H^1\left((0, 1), \Lambda\right), \qquad \mathcal{X} = H^1\left((0, 1), \bar{\Lambda}\right),$$

where $\bar{\Lambda} = \{x \in \mathbb{R}^{Nd} \mid \sum_{i=1}^n m_i x_i = 0\}$ is the closure of the set Λ defined in (6). The elements of Ω will be termed *collisionless orbits* and their center of mass lies at the origin at every time. Since each element of Ω is a continuous function, it follows from standard arguments that the action functional $\mathcal{A}_N : \Omega \to \mathbb{R}$ is smooth. Moreover, critical points of $\mathcal{A}_N$ inside the open set Ω are collisionless, classical solutions of (1). It is clear that, in general, it is impossible to extend the definition of $\mathcal{A}_N$ to $\mathcal{X}$, and it is precisely this fact that prevents us from using standard tools for studying colliding solutions to (1).

In this paper we will take into account colliding solutions of (1) with finite action and isolated collision. More precisely, we introduce a class of "good" colliding solutions.

Definition 3.1. *A one-collision solution of (1), is a map $x \in C\left([0, 1], \mathbb{R}^{Nd}\right) \cap C^2\left((0, 1), \mathbb{R}^{Nd}\right)$ such that*

1. $x^{-1}(\Delta) \cap [0,1] = \{1\}$, *i.e. no other collisions take place in the time interval* $[0,1)$;

2. x *solves pointwise the system* (1) *in the interval* $[0,1)$.

3. $\mathcal{A}_N(x) < +\infty$;

Remark 3.2. *Roughly speaking, a one-collision solution tends to $\partial\Lambda$ as, and only as, $t \to 1$. Conditions 1 and 3 are strictly related (see for instance [5, 22]), we impose both not to enter in the details of this matter.*

Definition 3.3. *Let* $\mathbf{n} := \{1, 2, \ldots, N\}$. *A* colliding cluster *for a one-collision solution* x *is a subset* $\mathbf{k} \subset \mathbf{n}$ *such that*

1. $x_i(1) = x_j(1)$ *for all indices* $i \neq j$ *in* $\mathbf{k}$;

2. $x_i(1) \neq x_j(1)$ *for all* $i \in \mathbf{k}$ *and* $j \in \mathbf{n} \setminus \mathbf{k}$.

A collision will be termed total *if its associated cluster* $\mathbf{k} = \mathbf{n}$.

The main property of a one-collision solution x is that the action $\mathcal{A}_N$ has directional derivatives at x along compactly–supported directions. This allows us to consider x as a "critical point" of $\mathcal{A}_N$. The proof of the next lemma follows trivially from Definition 3.1.

Lemma 3.4. *Let* x *be a one-collision solution of* (1). *Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{A}_N(x + \varepsilon\psi) = 0, \quad \forall \psi \in C_0^\infty((0,1)).$$

Consider a one-collision solution $\bar{x}$ with a colliding cluster $\mathbf{k} \subset \mathbf{n}$. Without loss of generality, we can assume $\mathbf{k} = \{1, 2, \ldots, k\}$, so that the $N - k$ last components $(\bar{x}_{k+1}, \ldots, \bar{x}_N)$ of the one-collision solution $\bar{x}$ are kept fixed. We define the restriction of the action functional

$$\mathcal{A}_{N,\mathbf{k}}(x) = \mathcal{A}_k(x) + \int_0^1 \Big[\sum_{j=k+1}^N |\dot{\bar{x}}_j|^2 + \frac{1}{2} \sum_{\substack{j_1,j_2=k+1 \\ j_1 \neq j_2}}^N U_{j_1 j_2}(\bar{x}_{j_1} - \bar{x}_{j_2}) + \frac{1}{2} \sum_{i=1}^k \sum_{j=k+1}^N U_{ji}(\bar{x}_j - x_i) \Big],$$

for every $x \in H^1((0,1), \mathbb{R}^{kd})$ and

$$\mathcal{A}(x) := \mathcal{A}_k(x) + \int_0^1 W(t,x)dt,$$

where

$$W(t,x) := \frac{1}{2} \sum_{i=1}^k \sum_{j=k+1}^N U_{ji}(\bar{x}_j(t) - x_i) \tag{14}$$

is defined on $[0,1] \times \mathbb{R}^{kd}$. Since we have supposed that the action is finite at one-collision solutions the term $\int_0^1 \frac{1}{2} \sum U_{j_1 j_2}(\bar{x}_{j_1} - \bar{x}_{j_2})$ is finite (and constant), hence $\mathcal{A}_{N,\mathbf{k}}$ and $\mathcal{A}$ differ only by a constant. Of course, when all the bodies collide the two functionals coincide on $H^1((0,1), \mathbb{R}^{kd})$. In the sequel we will deal with the functional $\mathcal{A}$.

Remark 3.5. *Since at $t = 1$ the bodies in the cluster $\mathbf{k}$ do not collide with those in $\mathbf{n} \setminus \mathbf{k}$ (and no other collision occurs in $[0,1)$), there exists an open set $\mathcal{U} \subset \mathbb{R}^{kd}$ such that $(\bar{x}_1([0,1]), \ldots, \bar{x}_k([0,1])) \subset \mathcal{U}$ and $W \in C^2([0,1] \times \mathcal{U})$.*

For simplicity, we will write $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k)$. Indeed, the terms involving the remaining components are of class C^2. We define the radial and "angular" variables in the colliding cluster $\mathbf{k}$

$$r := |\bar{x}| = I^{\frac{1}{2}}(\bar{x}) \in \mathbb{R} \quad s := \frac{\bar{x}}{|\bar{x}|}. \tag{15}$$

Since we are dealing with a total collision solution, the following condition on the variable r holds:

$$\lim_{t \to 1} r(t) = 0 \text{ and } r(t) \neq 0, \quad \forall t \in [0,1). \tag{16}$$

Condition (16) means that the particles in $\mathbf{k}$ collide in their center of mass when $t = 1$ and they do not have any other collisions in the interval $[0,1)$. Since

$$\bar{x} = rs, \quad \dot{\bar{x}} = \dot{r}s + \dot{s}r, \quad |\dot{\bar{x}}|^2 = \dot{r}^2 + r^2|\dot{s}|^2,$$

we can write the action functional at $x = \bar{x}$ in terms of the new variables (r, s) as

$$\mathcal{A}(r, s) := \int_0^1 \frac{1}{2} r^{-(2+\alpha)} \left(r^{\frac{2+\alpha}{2}} \dot{r} \right)^2 + r^{-\alpha} \left(\frac{1}{2} |r^{\frac{2+\alpha}{2}} \dot{s}|^2 + U(s) + r^\alpha W(t, rs) \right) dt \tag{17}$$

We consider the time scaling

$$dt = r^{\frac{2+\alpha}{2}} d\tau, \tag{18}$$

and in the sequel we will note with a dot "$\cdot$" the derivative with respect to the variable t and with a prime "$\prime$" the one with respect to τ. Replacing (18) in (17) we then obtain

$$\mathcal{A}(r, s) = \int_0^{\tau^*} \frac{1}{2} \left(r^{-\frac{2+\alpha}{4}} r' \right)^2 + r^{\frac{2-\alpha}{2}} \left(\frac{1}{2} |s'|^2 + U(s) + r^\alpha W \left(\int_0^\tau r^{\frac{2+\alpha}{2}}, rs \right) \right) d\tau, \tag{19}$$

where

$$\tau^* = \int_0^1 r^{-\frac{2+\alpha}{2}} dt.$$

In equation (19) we make the variable change

$$\rho = r^{\frac{2-\alpha}{4}}, \quad \rho' = \frac{2-\alpha}{4} r^{-\frac{2+\alpha}{4}} r' \tag{20}$$

to obtain the action functional depending on (ρ, s)

$$\mathcal{A}(\rho, s) = \int_0^{\tau^*} \frac{1}{2} \left(\frac{4}{2-\alpha} \right)^2 (\rho')^2 + \rho^2 \left(\frac{1}{2} |s'|^2 + U(s) + \rho^{\frac{4\alpha}{2-\alpha}} W \left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right) \right) d\tau, \tag{21}$$

where

$$\beta := \frac{2(2+\alpha)}{2-\alpha} > 2. \tag{22}$$

Remark 3.6. *We notice that a set of quite similar variables were used in [13] to study the dynamical system (1) from a geometrical viewpoint. As far as we know, the use of these coordinate in a variational setting is new.*

In the variables (ρ, s, τ), the Euler–Lagrange equations read

$$-\left(\frac{4}{2-\alpha} \right)^2 \rho'' + \rho \left(|s'|^2 + 2U_\alpha(s) \right) + \beta \rho^{\beta-1} \int_\tau^{\tau^*} \left[\rho^\beta \nabla_t W \left(\int_0^u \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right) \right] du$$

$$+ \beta \rho^{\beta-1} W \left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right) + \frac{4}{2-\alpha} \rho^{\frac{3}{2}\beta} \nabla W \left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right) \cdot s = \lambda_1 \rho^{\beta-1}, \tag{23}$$

$$-2\rho\rho's' - \rho^2 s'' + \rho^2 \nabla U_{\alpha|\mathcal{E}}(s) + \rho^{2\frac{4+\alpha}{2-\alpha}} \nabla_t W\left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}}s\right) = \lambda_2 s. \tag{24}$$

It will be useful to have a more explicit formula for the Lagrange multipliers λ_1 and λ_2. We suppose here $W = 0$, since the computation is exactly the same in the general case. First of all, we observe that the total (constant) energy h (see (5)) can be written as

$$h = \frac{1}{\rho^\beta}\left(\frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2\left(\frac{1}{2}|s'|^2 - U_\alpha(s)\right)\right).$$

Therefore, from (23)

$$\frac{d}{d\tau}\left(\frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \frac{\lambda_1}{\beta}\rho^\beta\right) = \left(\left(\frac{4}{2-\alpha}\right)^2 \rho'' + \lambda_1 \rho^{\beta-1}\right)\rho'$$

$$= \rho\left(|s'|^2 + 2U_\alpha(s)\right)\rho' = \frac{d}{d\tau}\left(\frac{\rho^2}{2}\right)\left(|s'|^2 + 2U_\alpha(s)\right),$$

which implies that

$$\frac{d}{d\tau}\left(\rho^\beta\left(h_\alpha + \frac{\lambda_1}{\beta}\right)\right) = \frac{d}{d\tau}\left(\frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \frac{\lambda_1}{\beta}\rho^\beta\right) + \frac{d}{d\tau}\left(\frac{\rho^2}{2}\left(|s'|^2 - 2U_\alpha(s)\right)\right)$$

$$= |s'|^2 \frac{d}{d\tau}\rho^2 + \frac{\rho^2}{2}\frac{d}{d\tau}\left(|s'|^2 - 2U_\alpha(s)\right)$$

$$= 2\rho\rho'|s'|^2 + \rho^2 s'\cdot s'' - \rho^2\nabla U_{\alpha|\mathcal{E}}(s)\cdot s' = 0.$$

Hence the constant $h_\alpha + \frac{\lambda_1}{\beta}$ must be zero, i.e.

$$\lambda_1 = -\beta h_\alpha. \tag{25}$$

As for λ_2, we take the inner product of (24) with s and deduce immediately

$$\lambda_2 = \rho^2|s'|^2, \tag{26}$$

since $|s|^2 = 1$, $s\cdot s' = 0$ and $s\cdot s'' = -|s'|^2$. The next result describes the behavior of the new variables (ρ, s, τ) and of the potential U near the collision time. We do not give a proof here, but we refer to [4] for a variational proof of these results, which were already proved in a different way ([11, 22, 23]).

Proposition 3.7 (Asymptotic estimates). *Let x be a one-collision solution, $\mathbf{k}$ its colliding cluster and ρ, s, τ be defined by (15), (18), (20) The following properties hold true:*

(a) $\tau^* = +\infty$;

(b) There exists $b > 0$ such that $\displaystyle\lim_{\tau\to+\infty}\frac{\rho'}{\rho} = -\frac{2-\alpha}{4}\sqrt{2b}$;

(c) $\displaystyle\lim_{\tau\to+\infty} U(s(\tau)) = b$;

(d) $\displaystyle\lim_{\tau\to+\infty}\operatorname{dist}\left(\mathcal{C}^b, s(\tau)\right) = \lim_{\tau\to+\infty}\inf_{\bar{s}\in\mathcal{C}^b}|s(\tau) - \bar{s}| = 0$, where

$$\mathcal{C}^b := \left\{s : U(s) = b, \nabla U_{|\mathcal{E}}(s) = 0\right\}.$$

is the set of central configurations (for the potential U) at level b;

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(e) $\int_0^\infty \frac{\rho'}{\rho}|s'|^2 < +\infty$.

(f) $\lim_{\tau \to +\infty} |s'(\tau)| = 0$.

Point *(d)* of Proposition 3.7 does not mean that the variable s converges to an element of the set $\mathcal{C}^b$; in this section we will deal with those collision solutions that admit a limiting central configuration. This fact is expressed as follows.

Definition 3.8. *We say that a one-collision solution x is* asymptotic to a central configuration s_0 *if* $\lim_{\tau \to +\infty} s(\tau) = s_0$. *More generally, we say that the one-collision solution x is* asymptotic to the set of central configurations $\mathcal{C}^b$ *if (d) in Proposition 3.7 is verified.*

4   A class of colliding motions with non–trivial Morse index

We now introduce a notion of Morse index for one collision solutions with respect to the angular variable s. The idea is to use the fact that in the new coordinates set $(\tau, \rho$ and $s)$ the collision take place at $+\infty$.

Lemma 4.1. *Let x be a one-collision solution, and let $v \in T_s\mathcal{E}$ be a compactly–supported function. Then*

$$\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)(v, v) = \int_0^{+\infty} \rho^2 \left[|v'|^2 + \nabla^2 U_{|\mathcal{E}}(s)(v, v)\right] + \rho^{2\frac{6+\alpha}{2-\alpha}} \frac{\partial^2}{\partial x^2} W\left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s\right)(v, v)\, d\tau. \quad (27)$$

Proof. The proof relies on very standard arguments. Take formula (21) and observe that the term

$$\int_0^{+\infty} \frac{1}{2}\left(\frac{4}{2-\alpha}\right)^2 (\rho')^2 + \rho^2 \frac{|s'|^2}{2}\, d\tau$$

represents $\int_0^1 |\dot{x}|^2\, dt$, which is clearly smooth for $x \in H^1((0,1), \mathbb{R}^{kd})$. Therefore, we need to show that the functional

$$\Psi \colon (\rho, s) \in [0, +\infty) \times \mathcal{E} \mapsto \int_0^{+\infty} \rho^2 \left(U(s) + \rho^{\frac{4\alpha}{2-\alpha}} W\left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s\right)\right) d\tau,$$

is twice differentiable in the variable s along compactly supported directions. Tt follows at once from Proposition 3.7 that ρ decays exponentially fast and both $U(s(\tau))$ and $W(\int_0^\tau \rho^\beta, \rho^{4/(2-\alpha)}s)$ remain bounded as $\tau \to +\infty$. Hence we can apply the Dominated Convergence Theorem to show that

$$\begin{aligned}
\frac{\partial^2 \Psi}{\partial s^2}(\rho, s)(v, v) &= \left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \Psi\left(\rho, \frac{s + \varepsilon v}{\sqrt{I(s + \varepsilon v)}}\right) \\
&= \int_0^{+\infty} \rho^2 \nabla^2 U_{|\mathcal{E}}(s)(v, v) + \rho^{2\frac{6+\alpha}{2-\alpha}} \frac{\partial^2}{\partial x^2} W\left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s\right)(v, v)\, d\tau.
\end{aligned}$$

$\square$

Definition 4.2. *Let $x = \rho^{4/(2-\alpha)}s$ be a one-collision solution. We define the* collision Morse index $m_c = m_c(\mathcal{A}, \rho, s)$ *of $\mathcal{A}$ at (ρ, s) as the supremum of all integers m such that there exist m linearly independent functions $\psi_1, \ldots, \psi_m \in C_0^\infty(\mathbb{R}^+, T_s\mathcal{E})$ with the property that $\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)$ is negative definite on $\mathrm{span}\,\{\psi_1, \ldots, \psi_m\}$. More precisely, $\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)(v, v) < 0$ for all $v \in \mathrm{span}\,\{\psi_1, \ldots, \psi_m\}$. Moreover, we will also say that $\mathcal{A}_N$ has collision Morse index m_c at x.*

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Our aim is to show that, under a suitable assumption on the eigenvalues of the Hessian $\nabla^2 U_{|\mathcal{E}}(s_0)$, a one-collision solution asymptotic to s_0 cannot be locally minimal for the action functional (21). This is the content of the next theorem.

Theorem 4.3. *Let $x \in \mathcal{X}$ be a one-collision solution of (1) asymptotic to a central configuration s_0. Then the collision Morse index of $\mathcal{A}_n$ at (ρ, s) is infinite, provided that the smallest eigenvalue μ_1 of $\nabla^2 U_{|\mathcal{E}}(s_0)$ satisfies*

$$\mu_1 < -\frac{(2-\alpha)^2}{8} U(s_0). \tag{28}$$

Proof. We introduce the variables ρ, s and τ defined in (20), and according with Definition 4.2 we will show that there exist infinitely many linearly independent functions $w_1, w_2, \dots$ such that $\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)(w_i, w_i) < 0$ for every index i. For any smooth, compactly supported function v such that $v(\tau) \in T_{s(\tau)}\mathcal{E}$ for all $\tau \geq 0$, we set $w = \rho v$. Then $w' = \rho' v + \rho v'$ and

$$\rho^2 |v'|^2 = |w'|^2 + |\rho'|^2 |v|^2 - 2\rho' w' v = |w'|^2 + \left(\frac{\rho'}{\rho}\right)^2 |w|^2 - 2\frac{\rho'}{\rho} w w'. \tag{29}$$

In terms of w, equation (27) becomes

$$\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)(v, v) = \int_0^{+\infty} \left[|w'|^2 + \left(\frac{\rho'}{\rho}\right)^2 |w|^2 - 2\frac{\rho'}{\rho} w' w + \nabla^2 U_{|\mathcal{E}}(s)(w, w) \right.$$
$$\left. + \rho^{2\beta} \frac{\partial^2}{\partial x^2} W \left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right)(w, w) \right] d\tau. \tag{30}$$

Setting

$$Q(w) = \int_0^{+\infty} \left[|w'|^2 + \left(\frac{\rho'}{\rho}\right)^2 |w|^2 - 2\frac{\rho'}{\rho} w' w + \nabla^2 U_{|\mathcal{E}}(s)(w, w) \right.$$
$$\left. + \rho^{2\beta} \frac{\partial^2}{\partial x^2} W \left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right)(w, w) \right] d\tau.$$

we will prove that $Q < 0$ on a vector space of infinite dimension. Let $0 < \ell_1 < \ell_2$ be arbitrary numbers, and take a positive real function $\varphi \in C_0^\infty(\ell_1, \ell_2)$; let $\{\tau_n\}_n$ be a strictly increasing, divergent sequence of positive numbers. We define $w_n(\tau) = \varphi(\tau - \tau_n)\xi$, where $\xi \in T_{s_0}\mathcal{E}$ will be chosen hereafter. In particular $w_n \in C_0^\infty((\ell_1 + \tau_n, \ell_2 + \tau_n), T_{s_0}\mathcal{E})$. It follows from Proposition 3.7 that the following estimates hold:

$$\int_0^{+\infty} \rho^{2\beta} \frac{\partial^2}{\partial x^2} W \left(\int_0^\tau \rho^\beta, \rho^{\frac{4}{2-\alpha}} s \right)(w_n, w_n) d\tau \leq C_1 \int_{\ell_1+\tau_n}^{\ell_2+\tau_n} \rho^{2\beta} |w_n|^2 d\tau \leq C_1 \|\varphi\|_\infty e^{-C_2\tau_n},$$

$$\int_0^{+\infty} \frac{\rho'}{\rho} w_n w_n' d\tau = \int_{\ell_1+\tau_n}^{\ell_2+\tau_n} \frac{\rho'}{\rho} w_n w_n' d\tau = -\frac{2-\alpha}{4} \sqrt{2U(s_0)} \int_{\ell_1+\tau_n}^{\ell_2+\tau_n} w_n w_n' d\tau + o(1) = o(1)$$

as $n \to +\infty$. In a similar way,

$$\int_0^{+\infty} \left|\frac{\rho'}{\rho}\right|^2 |w_n|^2 d\tau = \frac{(2-\alpha)^2}{8} U(s_0) \int_{\ell_1+\tau_n}^{\ell_2+\tau_n} |w_n|^2 d\tau + o(1)$$

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Putting together these estimates, using the continuity of $\nabla^2 U$ at s_0 and the fact that x is asymptotic to the central configuration s_0, we get

$$Q(w_n) = \int_{\ell_1+\tau_n}^{\ell_2+\tau_n} \left(|w_n'|^2 + \frac{(2-\alpha)^2}{8} U(s_0)|w_n|^2 + \nabla^2 U_{|\mathcal{E}}(s_0)(w_n, w_n) \right) d\tau + o(1). \qquad (31)$$

We choose now $\xi \in T_{s_0}\mathcal{E}$ as a normalized eigenvalue of $\nabla^2 U_{|\mathcal{E}}(s_0)$ corresponding to the eigenvalue μ_1, verifying assumption (28). Then equation (31) becomes

$$Q(w_n) = \int_{\ell_1+\tau_n}^{\ell_2+\tau_n} \left[|w_n'|^2 + \left(\frac{(2-\alpha)^2}{8} U(s_0) + \mu \right) |w_n|^2 \right] d\tau + o(1).$$

Since *both* φ and its support (ℓ_1, ℓ_2) are arbitrary, it follows from (28) that $\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)(w_n, w_n) < 0$ for all $n \gg 1$. We can now repeat the same construction with different choices of φ and of the sequence $\{\tau_n\}_n$, and build a countable family of functions $\{w_n\}_n$ with disjoint supports and such that $\frac{\partial^2 \mathcal{A}}{\partial s^2}(\rho, s)(w, w) < 0$ for all $w \in \text{span}\{w_1, w_2, \dots\}$. From Definition 4.2 it follows that the collision Morse index of $\mathcal{A}$ at x is infinite. $\qquad \square$

In the next section we will present some concrete examples in which our Theorem 4.3 applies.

5   Applications of Theorem 4.3

In this section we discuss the applicability of Theorem 4.3 to concrete examples of limiting central configurations. Clearly, the hardest assumption to check is inequality (28). Since it is known that the smallest eigenvalue μ_1 of $\nabla^2 U_{|\mathcal{E}}(s_0)$ at the central configuration s_0 is characterized by

$$\mu_1 = \min \left\{ \nabla^2 U_{|\mathcal{E}}(s_0)(v, v) \mid v \in T_{s_0}\mathcal{E}, \ \sum_i m_i v_i = 0, \ \|v\| = 1 \right\}, \qquad (32)$$

using (8) we obtain that (28) is implied by the existence of a vector $v \in T_{s_0}\mathcal{E}$ such that $\sum_i m_i v_i = 0$, $\|v\| = 1$ and

$$\nabla^2 U(s_0)(v, v) + \alpha U(s_0)\langle Mv, v \rangle < -\frac{(2-\alpha)^2}{8} U(s_0). \qquad (33)$$

In particular, when all the masses are equal to 1, we obtain the simpler condition

$$\nabla^2 U_{|\mathcal{E}}(s_0)(v, v) < -\frac{(2-\alpha)^2}{8} U(s_0). \qquad (34)$$

When each v_j is assumed to be orthogonal to the vector space generated by the configuration s_0 (see (11)) we can introduce the square matrix A, defined in (12). Hence (33) and (34) are satisfied provided we can find a vector $v = (v_1, \dots, v_N) \in \mathbb{R}^N$, such that $\|v\| = 1$ and

$$-\langle v, MAv \rangle + U(s_0)\langle v, Mv \rangle < -\frac{(2-\alpha)^2}{8\alpha} U(s_0) \qquad (35)$$

and

$$\langle w, Aw \rangle > \frac{(\alpha+2)^2}{8\alpha} U(s_0), \qquad (36)$$

respectively. We will prove that for a wide range of values of α (including the value $\alpha = 1$) the collinear central configuration of three equal masses and the regular N-gon configuration satisfy (28), showing (36). In particular, in the second case, when N is even, we will prove that (36) is satisfied for a vector $w \in \mathbb{R}^N$ that verifies the hip-hop symmetry (see [6] and [25]).

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5.1 Collinear central configurations for three equal masses

We consider the collinear central configuration of three particles of masses $m_1 = m_2 = m_3 = 1$, lying on a straight line

$$s_0 = \left((-1/\sqrt{2}, 0), (0,0), (1/\sqrt{2}, 0) \right).$$

We perform a planar variation as follows:

$$v = \left((\cos\theta, \sin\theta), (0, -2\sin\theta), (-\cos\theta, \sin\theta) \right),$$

where $\theta \in [0, \pi/2]$. We remark that $v = (v_1, v_2, v_3) \in T_{s_0}\mathcal{E}$, $\sum_{i=1}^3 v_i = 0$, and $\|v\|^2 = 2(1 + 2\sin^2\theta)$. With these choices the Hessian at the configuration s_0 is (see (10))

$$\nabla^2 U(s_0)(v, v) = 2\alpha \left\{ \cos^2\theta \left[2(\alpha+10)2^{\alpha/2} + (\alpha+1)2^{-\alpha/2} \right] - 18 \cdot 2^{\alpha/2} \right\}$$

Therefore, after dividing out by $\|v\|^2$, (28) reads

$$\frac{1}{1 + 2\sin^2\theta} \left\{ \cos^2\theta \left[2(\alpha+10)2^{\alpha/2} + (\alpha+1)2^{-\alpha/2} \right] - 18 \cdot 2^{\alpha/2} \right\}$$

$$< -\frac{(\alpha+2)^2}{8\alpha} \left(2 \cdot 2^{\alpha/2} + 2^{-\alpha/2} \right) \quad (37)$$

It is apparent that the most convenient choice, in order to get the widest range of α's, is $\theta = \pi/2$, i.e. to take normal variations. Hence (37) reduces to

$$\frac{6 \cdot 2^\alpha}{2 \cdot 2^\alpha + 1} > \frac{(\alpha+2)^2}{8\alpha} \quad (38)$$

Let $f(\alpha) := \frac{6 \cdot 2^\alpha}{2 \cdot 2^\alpha + 1}$ and $g(\alpha) := \frac{(\alpha+2)^2}{8\alpha}$ be respectively the left and right hand side of (38); since g strictly decreases on $(0, 2]$, f strictly increases $[0, 2]$ and $f(0) = g(6 - 4\sqrt{2})$, we conclude the existence of $\bar{\alpha} < 6 - 4\sqrt{2}$ such that for every $\alpha \in [\bar{\alpha}, 2]$ the inequality (38) holds true.

In a similar way, we can consider the central configuration of three masses $m_1 = m_3$ but m_2 different. Indeed, in this case we have a central configuration s_0 whose points are

$$s_0 = \left((-1/\sqrt{2m_1}, 0), (0, 0), (1/\sqrt{2m_1}, 0) \right).$$

We then choose again the normal variation

$$v_1 = (0, 1), \quad v_2 = (0, -2), \quad v_3 = (0, 1)$$

and observe that condition (35) reads now

$$\frac{9 \cdot 2^{\frac{\alpha+2}{2}} m_1^{\frac{\alpha+4}{2}} m_2 - (m_1 + 2m_2) \left(2^{\frac{\alpha+2}{2}} m_1^{\frac{\alpha+2}{2}} m_2 + 2^{-\frac{\alpha}{2}} m_1^{\frac{\alpha+4}{2}} \right)}{2^{\frac{\alpha+2}{2}} m_1^{\frac{\alpha+2}{2}} m_2 + 2^{-\frac{\alpha}{2}} m_1^{\frac{\alpha+4}{2}}} > \frac{3(2-\alpha)^2}{8\alpha}$$

After some simplifications, this is equivalent to the inequality

$$\frac{9 \cdot 2^{\alpha+1} m_1 m_2 - (m_1 + 2m_2)(2^{\alpha+1} m_2 + m_1)}{2^{\alpha+1} m_2 + m_1} > \frac{3(2-\alpha)^2}{8\alpha}.$$

Since this inequality is homogeneous with respect to the masses m_1 and m_2, we can suppose now $m_2 = 1$. Hence we should solve

$$\frac{2^\alpha(16m_1 + 4) - m_1^2 + 2m_1}{2^{\alpha+1} + m_1} > \frac{3(2-\alpha)^2}{8\alpha}. \quad (39)$$

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Set now
$$f(\alpha) = \frac{2^\alpha(16m_1 + 4) - m_1^2 + 2m_1}{2^{\alpha+1} + m_1}, \quad g(\alpha) = \frac{3(2-\alpha)^2}{8\alpha}.$$

One checks easily that $g(2) = 0$ and g is a positive, strictly decreasing function on $(0,2)$. Moreover, since
$$\frac{f'(\alpha)}{2^\alpha \log 2} = \frac{18m_1^2}{(2^{\alpha+1} + m_1)^2},$$

the function f is strictly increasing to the value $f(2) = \frac{-m_1^2 + 66m_1 + 16}{m_1 + 8}$. We conclude that we can find a number $\alpha^* > 0$ such that (39) is satisfied for all $\alpha > \alpha^*$ if and only if $f(2) > g(2)$, i.e. $m_1^2 - 66m_1 - 16 < 0$, or $m_1 < 33 + \sqrt{33^2 + 16}$. The Newtonian case $\alpha = 1$ is admissible if and only if $f(1) > g(1)$, i.e. $\frac{2(16m_1 + 4) - m_1^2 + 2m_1}{4 + m_1} > 3/8$, or $8m_1^2 - 269m_1 + 52 > 0$. Hence $m_1 < 34$ is enough.

Remark 5.1. *For the collinear configuration of three equal masses we can try to verify* (28) *instead of the stronger* (34). *Observe that the vector* $(1,1,1)$ *is an eigenvector for A with eigenvalue* 0. *Hence we restrict the matrix A to the space orthogonal to this vector that is $Y = \{v = (v_1, v_2, v_3) : \sum_i v_i = 0\}$ spanned by*
$$w_1 = (1, 0, -1), \quad w_2 = (0, 1, -1).$$

If $B = [b_{hk}]$ denotes the symmetric matrix A restricted to the space Y we have that
$$b_{hk} = w_h^T A w_k = a_{hk} - (a_{h3} + a_{k3}) + a_{33} = b_{kh},$$

where w^T denotes transposition of the vector w. Explicitly,
$$B = \begin{pmatrix} 2\gamma + 4\gamma^{-1} & \gamma + 2\gamma^{-1} \\ \gamma + 2\gamma^{-1} & 5\gamma + \gamma^{-1} \end{pmatrix}, \quad \text{with } \gamma = 2^{\frac{\alpha+2}{2}},$$

and its eigenvalues are
$$\lambda_B^{1,2} = \frac{7\gamma + 5\gamma^{-1} \pm \sqrt{13\gamma^2 - 2 + 25\gamma^{-2}}}{2}.$$

It is easy to check that condition (36) *is implied by the inequality*
$$\frac{7\gamma + 5\gamma^{-1} + \sqrt{13\gamma^2 - 2 + 25\gamma^{-2}}}{2} > \frac{(2+\alpha)^2}{8\alpha}\left(\gamma + 2\gamma^{-1}\right). \tag{40}$$

This approach gives a wider range of "good" values for the parameter α (as we can see in Figure 1), but is clearly impossible to apply in more general situations.

5.2 The regular N-gon central configuration

We now consider the case of a planar central configuration of $N \geq 4$ equal bodies with equal masses, lying at the vertices of a regular N-gon inscribed in a circle of radius $1/\sqrt{N}$. In the sequel, we will systematically use the notation
$$r_{ij} = r_{ij}(N) = \frac{2}{\sqrt{N}} \sin\left(\frac{|i-j|}{N}\pi\right)$$

for the distance between the i-th and the j-th bodies. We remark that $r_{ij} = r_{1k}$, where $k = |i-j|+1$.

The aim of this section is to show that when s_0 is the polygonal central configuration, then relation (28) is verified for a whole interval of α's including the Newtonian case $\alpha = 1$. This will prove that collisions ending up in a polygonal configuration cannot be minima for the action.
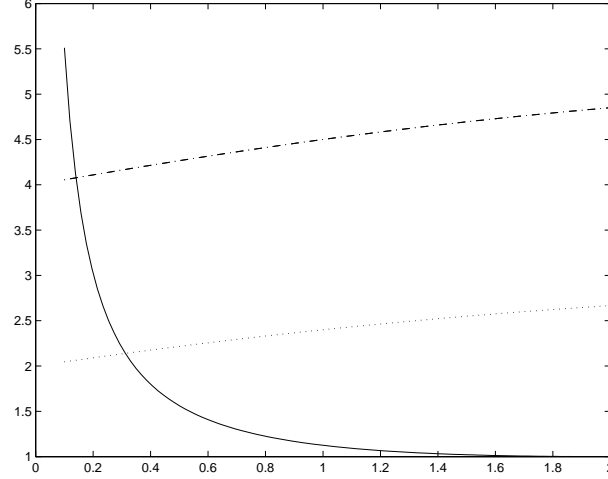
Figure 1: Validity of (38) in the dotted line compared with the validity of (40). The continue line represents the quantity $\frac{(2+\alpha)^2}{8\alpha}$.

Condition (36) in this case reads

$$\sum_{i,j} a_{ij} w_i w_j > \frac{(\alpha+2)^2}{8\alpha} U(s_0) = \frac{(\alpha+2)^2}{8\alpha} \frac{N}{2} \sum_{k=2}^{N} \frac{1}{r_{1k}^{\alpha}}. \tag{41}$$

We rewrite (41) in the form

$$\Psi_N(\alpha) := \frac{2}{N} \frac{\sum_{i,j} a_{ij} w_i w_j}{\sum_{k=2}^{N} r_{1k}^{-\alpha}} > \frac{(\alpha+2)^2}{8\alpha}. \tag{42}$$

Define, for simplicity, $\tilde{r}_{ij} = \frac{\sqrt{N}}{2} r_{ij} = \sin\left(\frac{|j-i|}{N}\pi\right)$, $\tilde{a}_{ij} = \left(\frac{\sqrt{2}}{N}\right)^{\alpha+2} a_{ij}$, so that the matrix $\tilde{A} = [\tilde{a}_{ij}]$ can be constructed by writing $\tilde{r}_{ij}$ instead of r_{ij} in (12). Observe that

$$\Psi_N(\alpha) = \frac{1}{2} \frac{\sum_{i,j} \tilde{a}_{ij} w_i w_j}{\sum_{k=2}^{N} \tilde{r}_{1k}^{-\alpha}} = \frac{1}{2} \frac{\sum_{i=1}^{N} \tilde{a}_{ii} w_i^2 + 2\sum_{i<j} \tilde{a}_{ij} w_i w_j}{\sum_{k=2}^{N} \tilde{r}_{1k}^{-\alpha}}.$$

We choose w as follows:

$$w = \begin{cases} (1/2, -1/2, 1/2, -1/2) & \text{for } N = 4 \\ (1/\sqrt{2}, -1/\sqrt{2}, 0, \ldots, 0) & \text{for } N \geq 5. \end{cases} \tag{43}$$

Remark 5.2. *Equivalently, when $N \geq 5$, we could choose w such that $w_i = 1/\sqrt{2}$, $w_{i+1} = -1/\sqrt{2}$, and $w_k = 0$ for any $k \notin \{i, i+1\}$. We will use this observation later on.*

Therefore

$$\Psi_4(\alpha) = \Phi_4(\alpha) + \frac{1}{2} \frac{2\tilde{r}_{12}^{-\alpha-2} - \tilde{r}_{13}^{-\alpha-2}}{\sum_{k=2}^{4} \tilde{r}_{1k}^{-\alpha}}$$

$$\Psi_N(\alpha) = \Phi_N(\alpha) + \frac{1}{2} \frac{\tilde{r}_{12}^{-\alpha-2}}{\sum_{k=2}^{N} \tilde{r}_{1k}^{-\alpha}} \qquad (N \geq 5),$$

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where

$$\Phi_N(\alpha) := \frac{1}{2}\frac{\sum_{k=2}^{N} \tilde{r}_{1k}^{-\alpha-2}}{\sum_{k=2}^{N} \tilde{r}_{1k}^{-\alpha}}, \quad \text{for all } N \geq 4. \tag{44}$$

We now state and prove some technical lemmata that are useful for the proof of the main theorem of this section. The first one is a simple exercise in first year calculus.

Lemma 5.3. *Let $\{b_j\}_{j=1,\dots,n}$ be a family of n positive real numbers such that*

$$b_1 > b_2 > \dots > b_n > 1.$$

Then the two functions

$$f(x) := \frac{\sum_{j=1}^{n} b_j^{x+2}}{\sum_{j=1}^{n} b_j^{x}} \quad \text{and} \quad g(x) := \frac{1 + \sum_{j=1}^{n} b_j^{x+2}}{1 + \sum_{j=1}^{n} b_j^{x}}$$

are strictly increasing on the interval $[0, 2]$.

Proof. One computes the first derivatives of f and g by the rule for a quotient. Then the sign of these derivatives is the sign of the numerator. When dealing with f, after simplifying some terms, we end up with a sum of terms like

$$b_i^x b_j^x (b_i^2 - b_j^2)(\log b_i - \log b_j), \quad (i < j)$$

and these are all positive, because of the monotonicity of the family $\{b_j\}$. When dealing with g, an extra term appears due to the presence of the number 1. Nevertheless, we easily check that the extra term is just

$$\sum_j b_j^x (b_j^2 - 1) \log b_j,$$

which is positive since $b_j > 1$. $\qquad\square$

Lemma 5.4. *For every $N \geq 4$, the function $\Phi_N \colon [0,2] \to \mathbb{R}$ is strictly increasing and satisfies*

$$\Phi_N(0) > \frac{N-1}{N}. \tag{45}$$

Proof. The monotonicity of Φ_N follows easily from Lemma 5.3, by exploiting the symmetry of the regular N-gon with respect to a straight line passing through a fixed vertex. If N is odd, one has

$$\Phi_N(\alpha) := \frac{1}{2}\frac{\sum_{k=2}^{(N+1)/2} \tilde{r}_{1k}^{-\alpha-2}}{\sum_{k=2}^{(N+1)/2} \tilde{r}_{1k}^{-\alpha}},$$

where $0 < \tilde{r}_{1k} = \sin\left(\frac{k-1}{N}\pi\right) \leq \sin\left(\frac{N-1}{N}\pi\right) < 1$. Now we can use the monotonicity of f. If N is even one uses the monotonicity of g with $a_j := \sin\left(\frac{j}{N}\pi\right)$, $j = 1, \dots, \left[\frac{N+1}{2}\right] - 1$. The proof of (45) is equivalent to the following inequality

$$\sum_{k=2}^{N} \tilde{r}_{1k}^{-2} > \frac{2(N-1)^2}{N}$$

Since $t \mapsto t^{-2}$ is a convex function on $(0, \infty)$, the discrete Jensen inequality tells us that

$$\left(\frac{1}{N-1}\sum_{k=2}^{N} \tilde{r}_{1k}\right)^{-2} \leq \frac{1}{N-1}\sum_{k=2}^{N} \tilde{r}_{1k}^{-2}, \quad \text{i.e.}$$

$$\frac{(N-1)^3}{\left(\sum_{k=2}^{N}\tilde{r}_{1k}\right)^2} \le \sum_{k=2}^{N}\tilde{r}_{1k}^{-2}.$$

Hence the following inequality implies (45)

$$\frac{(N-1)^3}{\left(\sum_{k=2}^{N}\tilde{r}_{1k}\right)^2} > \frac{2(N-1)^2}{N}, \quad \text{or}$$

$$\sum_{k=2}^{N}\tilde{r}_{1k} < \sqrt{\frac{N(N-1)}{2}}.$$

Since $\tilde{r}_{1k} = \operatorname{Im} e^{i\frac{k-1}{N}\pi}$, we can easily compute

$$\sum_{k=2}^{N}\tilde{r}_{1k} = \operatorname{cotan}\frac{\pi}{2N} = \frac{\sin\frac{\pi}{N}}{1-\cos\frac{\pi}{N}}$$

Therefore, we have to prove that

$$\frac{\sin\frac{\pi}{N}}{1-\cos\frac{\pi}{N}} < \sqrt{\frac{N(N-1)}{2}}, \qquad \text{for all } N \ge 4.$$

We set $x = \pi/N$, so that the last inequality reads

$$\frac{\sin x}{1-\cos x} < \sqrt{\frac{\pi(\pi-x)}{2x^2}} = \frac{1}{x}\sqrt{\frac{\pi(\pi-x)}{2}}$$

We will now show that

$$\frac{x\sin x}{1-\cos x} < \sqrt{\frac{\pi(\pi-x)}{2}} \qquad \text{for all } x \in (0, \pi/4).$$

Set

$$f(x) = \frac{x\sin x}{1-\cos x}, \qquad g(x) = \sqrt{\frac{\pi(\pi-x)}{2}}.$$

We now prove that $g-f$ is strictly decreasing in the interval $(0, \pi/4)$ and that $g(\pi/4)-f(\pi/4) > 0$. More precisely, we claim that

$$g'(x) - f'(x) = \frac{\operatorname{cosec}^2\frac{x}{2}}{8\sqrt{\pi-x}}\left\{\sqrt{2\pi}(\cos x - 1) + 4(x - \sin x)\sqrt{\pi-x}\right\} < 0$$

All we have to show is that

$$\sqrt{2\pi}(\cos x - 1) + 4(x - \sin x)\sqrt{\pi-x} < 0$$

for all $x \in (0, \pi/4)$, or

$$\sqrt{2\pi}(\cos x - 1) < 4\sqrt{\pi-x}(\sin x - x).$$

But

$$\sin x - x > -\frac{x^3}{6}, \quad \cos x - 1 < -\frac{x^2}{2} + \frac{x^4}{4!}$$

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and this implies that we can check the inequality

$$\sqrt{2\pi}x^2\left(\frac{x^2}{4!}-\frac{1}{2}\right) < -\frac{2}{3}x^3\sqrt{\pi-x},$$

or

$$\sqrt{2\pi}(x^2-12) < -16x\sqrt{\pi-x}.$$

Both sides of this inequality are negative, since $x \in (0, \pi/4)$. We now square and reverse the sense of the inequality, and have to prove that

$$2\pi x^4 + 256x^3 - 304\pi x^2 + 288\pi > 0.$$

This is clearly true, since $2\pi x^4 + 256x^3 - 304\pi x^2 + 288\pi > 2\pi x^4 - 304\pi x^2 + 288\pi$ and this bi-quadratic equation has no real roots in $(0, \pi/4)$. To complete the proof, we need to show that $g(\pi/4) - f(\pi/4) > 0$. But

$$g(\pi/4) - f(\pi/4) = \frac{\pi}{2}\sqrt{\frac{3}{2}} - \frac{\pi}{4\sqrt{2}-1} > 0.$$

$\square$

Lemma 5.5. *For each fixed $N \geq 4$, the map $\Psi_N \colon [0, 2] \to \mathbb{R}$ is strictly increasing.*

Proof. With the same arguments contained in the proof of Lemma 5.3, one can prove that

$$\frac{2\tilde{r}_{12}^{-\alpha-2} - \tilde{r}_{13}^{-\alpha-2}}{\sum_{k=2}^4 \tilde{r}_{1k}^{-\alpha}} \quad \text{and} \quad \frac{\tilde{r}_{12}^{-\alpha-2}}{\sum_{k=2}^N \tilde{r}_{1k}^{-\alpha}} \qquad (N \geq 5)$$

are monotone functions of $\alpha \in (0, 2)$. In particular for the first one, we exploit the fact that $\tilde{r}_{12} = \tilde{r}_{14}$. $\square$

We conclude using Lemma 5.4.

Lemma 5.6. *For each $N \geq 4$, there results $\Psi_N(0) > 9/8$.*

Proof. When $N = 4$ we compute easily that $\Psi_4(0) > 5/4 > 9/8$. When $N = 5$ we use $\sin(\pi/5) = \sqrt{10-2\sqrt{5}}/4$ and easily verify that $\Psi_5(0) > \frac{6\sqrt{5}-1}{5(\sqrt{5}-1)} > 9/8$. For $N \geq 6$, we have to verify that $\Psi_N(0) = \Phi_N(0) + \frac{1}{2(N-1)}\frac{1}{\sin^2(\pi/N)}$. From (45) we have that this inequality is implied by

$$\frac{1}{\sin^2(\frac{\pi}{N})} > \frac{(N+8)(N-1)}{4N} \qquad \text{for all } N \geq 6.$$

To conclude, recall that $\sin\frac{\pi}{N} < \frac{\pi}{N}$. Hence

$$\frac{1}{\sin^2(\frac{\pi}{N})} > \frac{N^2}{\pi^2},$$

and it is simple to check that for all $N \geq 6$ there results

$$4N^3 > \pi^2 N^2 + 7\pi^2 N - 8\pi^2.$$

$\square$

Theorem 5.7. *For every $N \geq 4$, there exists $\alpha_N < 1$ such that (42) holds for every $\alpha \geq \alpha_N$.*

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Proof. Let $f(\alpha) = \frac{(\alpha+2)^2}{8\alpha}$ be the right-hand side of (42). This is a strictly decreasing function on the interval $(0,2)$. Since $f(1) = 9/8$, we deduce from lemmas 5.5 and 5.6 that the graphs of Φ_N and of f must intersect at a unique point $\alpha_N < 1$. This concludes the proof. $\qquad\square$

When N is even and greater than 4, we provide an example of a vertical variation that verifies (36) and also satisfies the *hip-hop* symmetry (in $\mathbb{R}^3$). Denoting $x_k = (\xi_k, z_k) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ the position of the k-th body, this symmetry constraint, studied in [6] and [25], imposes that

$$\forall k = 1, \ldots, N-1 \qquad \xi_{k+1}(t) = e^{\frac{\pi i}{N}} \xi_k(t), \quad z_{k+1}(t) = -z_k(t).$$

The simple variation on the N-gon configuration introduced in (43) is then no longer admissible as soon as $N \geq 6$; however, we can consider the equivariant vector w orthogonal to the plane of the central configuration whose "vertical" components are $w_i = (-1)^i/\sqrt{N}$, for $i = 1, 2, \ldots, N$. Inequality (41) is then equivalent to

$$\frac{1}{N} \sum_{i,j} (-1)^{i+j} a_{ij} > h_N(\alpha), \qquad \text{where} \quad h_N(\alpha) = \frac{(\alpha+2)^2}{8\alpha} \frac{N}{2} \sum_{k=2}^{N} \frac{1}{r_{1k}^{\alpha}}. \tag{46}$$

We already know (see Remark 5.2) that whenever we choose two consecutive bodies of the polygon (say the i-th and the $(i+1)$-th) and we take $w_i = -w_{i+1} = 1/\sqrt{2}$ then

$$\frac{1}{2} \left(a_{ii} + a_{i+1\,i+1} - 2a_{i\,i+1} \right) > h_N(\alpha).$$

In particular if we take $i = 1, 3, 5, \ldots, N-1$ and we sum the corresponding $N/2$ inequalities we obtain

$$\frac{1}{2} \left(\sum_{i=1}^{N} a_{ii} - 2 \sum_{i \in \{1,3,5,\ldots,N-1\}} a_{i\,i+1} \right) > \tfrac{N}{2} h_N(\alpha),$$

or

$$\frac{1}{N} \left(\sum_{i=1}^{N} a_{ii} - 2 \sum_{i=1,3,5,\ldots,N-1} a_{i\,i+1} \right) = \frac{1}{N} \left(N a_{11} - 2\frac{N}{2} a_{12} \right) > h_N(\alpha). \tag{47}$$

Comparing (46) and (47) we conclude if we can prove

$$-2\frac{N}{2} a_{12} - 2N \sum_{j=3}^{N/2} (-1)^j a_{1j} - 2\frac{N}{2} (-1)^{N/2+1} a_{1\,N/2+1} > 0$$

that is

$$-a_{12} - 2 \sum_{j=3}^{N/2} (-1)^j a_{1j} - (-1)^{N/2+1} a_{1\,N/2+1} > 0.$$

Replacing the expression of a_{ij} (always negative when $i \neq j$) we have to prove that

$$\frac{1}{r_{12}^{\alpha+2}} + 2 \sum_{j=3}^{N/2} \frac{(-1)^j}{r_{1j}^{\alpha+2}} + \frac{(-1)^{N/2+1}}{r_{1\,N/2+1}^{\alpha+2}} > 0$$

and recalling that $r_{ij}(N) = \frac{2}{\sqrt{N}} \sin\left(\frac{|i-j|}{N} \pi \right)$, inequality (46) will follow from

$$g(N, \alpha) = \frac{1}{\sin^{\alpha+2}\left(\frac{\pi}{N}\right)} + 2 \sum_{j=3}^{N/2} \frac{(-1)^j}{\sin^{\alpha+2}\left(\frac{j-1}{N}\pi\right)} + (-1)^{N/2+1} > 0. \tag{48}$$

where g is defined on the product $\{6, 8, 10, \ldots\} \times [0, 2]$. We then would like to prove (48) for every $N \geq 6$, N even, in an interval of values of α containing $\alpha = 1$. The sum of the first two terms of the function g is always positive, indeed

$$\frac{1}{\sin^{\alpha+2}\left(\frac{\pi}{N}\right)} - \frac{2}{\sin^{\alpha+2}\left(\frac{2\pi}{N}\right)} = \frac{\left[2\cos\left(\frac{\pi}{N}\right)\right]^{\alpha+2} - 2}{\sin^{\alpha+2}\left(\frac{2\pi}{N}\right)} = \left(\frac{2}{\sin\left(\frac{2\pi}{N}\right)}\right)^{\alpha+2}\left[\cos^{\alpha+2}\left(\frac{\pi}{N}\right) - \frac{1}{2^{\alpha+1}}\right]$$

and for every N greater then 6

$$\cos\left(\frac{\pi}{N}\right) > \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \geq 2^{-\frac{\alpha+1}{\alpha+2}}, \qquad \forall \alpha \in [0, 2].$$

The remaining terms can be collected in pairs of the kind

$$\frac{1}{\sin^{\alpha+2}\left(\frac{(j-1)\pi}{N}\right)} - \frac{1}{\sin^{\alpha+2}\left(\frac{j\pi}{N}\right)}, \qquad \text{with } j \text{ even}, \ 4 \leq j \leq \frac{N}{2}$$

whose sum is positive being $\sin x$ an increasing function when $x \in [0, \pi/2]$ and α positive. Concerning the other terms in g, two different situations can occur: when $N/2$ is even or when $N/2$ is odd. In the first case the last two terms of the function g are $2\sin^{-\alpha-2}\left(\frac{(N-2)\pi}{2N}\right)$ and -1 whose sum is strictly positive. When $N/2$ is odd there is just a positive remaining term in g which is $+1$.

6 Asymptotic minimality for the weak–force case

Equation (35) in Section 5 suggests that there should exist minimal colliding motions for small values of α. For the reader's sake, we will use a somehow more transparent notation to stress the dependence on the parameter α by writing U_α and $\mathcal{A}_\alpha$ instead of U and $\mathcal{A}$ defined in (2) and (3) respectively. Similarly $h_\alpha = \frac{1}{2}|\dot{x}_\alpha|^2 - U_\alpha(x_\alpha)$ denotes the energy of x_α, $\mathcal{C}_\alpha \subset \mathcal{E}$ the set of central configurations of U_α and we refer to $(1)_\alpha$ to recall the dynamical system (1) with the α-homogeneous potential U_α.

Throughout this section, we will consider *total* one-collision solutions (see Definition 3.3) x_α to $(1)_\alpha$ with the following "initial conditions" independent of α:

$$|\dot{x}_\alpha(0)| = |x_\alpha(0)| = 1.$$

In particular, the function W defined in (14) is identically zero. In Section 3 we have seen that there exists a diffeomorphism Ψ that rewrites any non-trivial $x \in H^1([0, 1], \mathbb{R}^{Nd})$ in the coordinates (ρ, s) and changes the time $t \in [0, 1]$ into a new time $\tau \in [0, +\infty)$. On the space of these new variables we will use the norm

$$\|(\zeta, v)\|_{\mathcal{H}}^2 = \int_0^{+\infty} \left(|\zeta'(\tau)|^2 + \zeta(\tau)^2\right) d\tau + \int_0^{+\infty} \left(|v'(\tau)|^2 + |v(\tau)|^2\right) d\tau$$

With an abuse of notation, we will continue to write $\mathcal{A}_\alpha$ instead of $\mathcal{A}_\alpha \circ \Psi$. Since the function W is identically zero, the Euler-Lagrange equations (23) and (24) reduce to

$$-\left(\frac{4}{2-\alpha}\right)^2 \rho'' + \rho\left[|s'|^2 + 2U_\alpha(s)\right] = -\beta h_\alpha \rho^{\beta-1}, \tag{49}$$

$$-2\rho\rho's' - \rho^2 s'' + \rho^2 \nabla U_{\alpha|\mathcal{E}}(s) = \rho^2 |s'|^2 s. \tag{50}$$

It is convenient to introduce some terminology.

Definition 6.1. *Let $x \in H^1([0,1], \mathbb{R}^{Nd})$ be given, and let s be the second component of $\Psi(x)$. A function φ is* compactly supported variation *on $(a,b) \subset (0,1)$ corresponding to x if $\varphi \in T_x H^1([0,1], \mathbb{R}^{Nd}) \simeq H^1([0,1], \mathbb{R}^{Nd})$ and if $v \colon [0,1] \to T_s \mathcal{E}$, where $\Psi(\varphi) = (\zeta, v)$.*

Furthermore, if $\Delta \subset \mathbb{R}^{Nd}$ is the set of collision configurations defined in (4), fixed $\delta > 0$ we term Δ^δ its open δ-neighborhood. The following lemma is crucial for the proofs of Theorems 6.4, 6.6 and 6.14.

Lemma 6.2. *Let $(x_\alpha)_{\alpha \in A}$, $A \subset (0,2)$, be a family of one-collision solutions for $(1)_\alpha$, $\alpha \in A$, and let $(K_\alpha)_{\alpha \in A}$, $K_\alpha \subset \mathcal{C}_\alpha$, be the sets of central configurations to which x_α is asymptotic. for every $\alpha \in A$, in the sense of Definition 3.8. If*

$$K_\alpha \subset \mathcal{C}_\alpha \setminus \Delta^\delta, \qquad \forall \alpha \in A \tag{51}$$

for some $\delta > 0$, then there exists $m > 0$, which does not depend on α, such that

$$\nabla^2 U_{\alpha|\mathcal{E}}(s_\alpha)(v,v) \geq -\alpha m |v|^2, \qquad \forall v \in T_{s_\alpha} \mathcal{E}$$

where $s_\alpha = x_\alpha / |x_\alpha|$, for every $\alpha \in A$.

Proof. The conclusion follows from equations (9) and (10) and the uniform assumption (51). $\square$

We recall some terminology.

Definition 6.3. *A solution $x_\alpha(t) = (x_{\alpha,1}, \ldots, x_{\alpha,N})(t)$ of $(1)_\alpha$ is called* homographic *if there exist two functions $\lambda(t) > 0$ and $A(t) \in SO(d)$ and a fixed configuration $\bar{s}_\alpha = (\bar{s}_{\alpha,1}, \ldots, \bar{s}_{\alpha,N})$ such that*

$$x_\alpha(t) = \lambda(t)(A(t)\bar{s}_{\alpha,1}, \ldots, A(t)\bar{s}_{\alpha,N})(t).$$

It is well known that the configuration s_α associated to a homographic motion is central, that is s_α is a critical point for the potential U constrained to the ellipsoid $\mathcal{E}$. See Definition 2.1 and Remark 2.2.

Let us state and prove the first result of this section.

Theorem 6.4. *Let $(x_\alpha)_{\alpha \in A}$, $A \subset (0,2)$, be a family of homographic one-collision solutions for $(1)_\alpha$ and let $\bar{s}_\alpha$ be such that $x_\alpha(t) = r_\alpha(t)\bar{s}_\alpha$, for all $t \in [0,1]$ and $\alpha \in A$. If there exists $\delta > 0$ such that*

$$(\bar{s}_\alpha)_\alpha \subset \mathcal{C}_\alpha \setminus \Delta^\delta \tag{52}$$

then there exists $\bar{\alpha} \in (0,2)$ such that for every $\alpha \in (0,\bar{\alpha}) \cap A$ there exists $t_\alpha = t(x_\alpha)$ such that[1]

$$\Delta \mathcal{A}_\alpha(x_\alpha, \varphi) := \mathcal{A}_\alpha(x_\alpha + \varphi) - \mathcal{A}_\alpha(x_\alpha) > 0 \tag{53}$$

for every compactly supported variation φ on $(t_\alpha, 1)$.

Furthermore if for every $\alpha \in A$ the energy of the homographic motion is positive then there exists $\bar{\bar{\alpha}} \in (0,2)$ such that for every $\alpha \in (0,\bar{\bar{\alpha}}) \cap A$ inequality (53) holds for every compactly supported variation φ on $(0,1)$.

Proof. We will prove (53) by switching to the new coordinates (ρ, s) and to the scaled time τ defined in (18). Since φ is a compactly supported variation and by virtue of (52), we see that $\mathcal{A}_\alpha = \mathcal{A}_\alpha \circ \Psi$ is smooth enough to write the Taylor expansion (where $(\zeta, v) = \Psi(\varphi)$ and supp $v \subset (0, +\infty)$)

$$\Delta \mathcal{A}_\alpha(x_\alpha, \varphi) = d^2 \mathcal{A}_\alpha(x_\alpha)((\zeta, v), (\zeta, v)) + o(\|(\zeta, v)\|_{\mathcal{H}}^2),$$

---

[1]With a slight abuse of notation, we write here $x_\alpha + \varphi$. This is justified by the fact that φ is a tangent vector at x_α to the linear space $H^1([0,1], \mathbb{R}^{Nd})$.

where the first order term disappears because x_α is a critical point of $\mathcal{A}_\alpha$. Therefore, it is sufficient to prove that $d^2\mathcal{A}_\alpha(x_\alpha)\left((\zeta,v),(\zeta,v)\right) > 0$ whenever α is small enough and the support of v is sufficiently away from 0. Equivalently, we will prove that

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho^2}(\zeta,\zeta) + 2\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho\partial s}(\zeta,v) + \frac{\partial^2\mathcal{A}_\alpha}{\partial s^2}(v,v) > 0, \tag{54}$$

where

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho^2}(\rho_\alpha,\bar{s}_\alpha)(\zeta,\zeta) \;=\; \int_0^{+\infty} \left(\frac{4}{2-\alpha}\right)^2 (\zeta')^2 + \zeta^2\left(|\bar{s}_\alpha'|^2 + 2U_\alpha(\bar{s}_\alpha)\right)\, d\tau, \tag{55}$$

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho\partial s}(\rho_\alpha,\bar{s}_\alpha)(\zeta,v) \;=\; 2\int_0^{+\infty} \rho_\alpha\zeta\left[\bar{s}_\alpha'\cdot v' + \nabla U_{\alpha|\mathcal{E}}(\bar{s}_\alpha)\cdot v\right]\, d\tau, \tag{56}$$

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha,\bar{s}_\alpha)(v,v) \;=\; \int_0^{+\infty} \rho_\alpha^2\left[|v'|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(\bar{s}_\alpha)(v,v)\right]\, d\tau. \tag{57}$$

Since we are dealing with homographic motions, $\bar{s}_\alpha$ is a constant function, critical point of U_α constrained to the ellipsoid $\mathcal{E}$, then $\dfrac{\partial^2\mathcal{A}_\alpha}{\partial\rho\partial s}(\rho_\alpha,\bar{s}_\alpha)(\zeta,v) = 0$ for every pair (ζ,v).

As in the proof of Theorem 4.3, we introduce the auxiliary variable $w = \rho_\alpha v$ and we wish to prove that

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho^2}(\rho_\alpha,\bar{s}_\alpha)(\zeta,\zeta) + \frac{\partial^2\mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha,\bar{s}_\alpha)(v,v)$$

$$= \int_0^{+\infty} \left\{\left(\frac{4}{2-\alpha}\right)^2 (\zeta')^2 + 2U_\alpha(\bar{s}_\alpha)\zeta^2 + \rho_\alpha^2\left[|v'|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(\bar{s}_\alpha)(v,v)\right]\right\}\, d\tau$$

$$= \int_0^{+\infty} \left\{\left(\frac{4}{2-\alpha}\right)^2 (\zeta')^2 + 2U_\alpha(\bar{s}_\alpha)\zeta^2 + |w'|^2 + \frac{\rho_\alpha''}{\rho_\alpha}|w|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(\bar{s}_\alpha)(w,w)\right\}\, d\tau > 0,$$

where in the last step we have integrated by parts. Using the Euler-Lagrange equation (49) divided by ρ we have

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho^2}(\rho_\alpha,\bar{s}_\alpha)(\zeta,\zeta) + \frac{\partial^2\mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha,\bar{s}_\alpha)(v,v)$$

$$= \int_0^{+\infty} \left\{\left(\frac{4}{2-\alpha}\right)^2 (\zeta')^2 + |w'|^2 + 2U_\alpha(\bar{s}_\alpha)\left[\zeta^2 + \left(\frac{2-\alpha}{4}\right)^2 |w|^2\right]\right\}\, d\tau$$

$$+ \int_0^{+\infty} \left\{\left(\frac{2-\alpha}{4}\right)^2 \beta h_\alpha \rho_\alpha^{\beta-2}|w|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(\bar{s}_\alpha)(w,w)\right\}\, d\tau.$$

Since U_α is positive, by the uniform assumption (52) and Lemma 6.2 there exist two positive constants C, m such that

$$\frac{\partial^2\mathcal{A}_\alpha}{\partial\rho^2}(\rho_\alpha,\bar{s}_\alpha)(\zeta,\zeta) + \frac{\partial^2\mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha,\bar{s}_\alpha)(v,v) \geq C\int_0^{+\infty} \left[(\zeta')^2 + \zeta^2 + |w'|^2\right]\, d\tau - \alpha m\int_0^{+\infty} |w|^2\, d\tau$$

$$+ \left(\frac{2-\alpha}{4}\right)^2 \int_0^{+\infty} \beta h_\alpha \rho_\alpha^{\beta-2}|w|^2\, d\tau.$$

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If we suppose $h_\alpha > 0$, for every $\alpha \in A$, we have that there exists $C_1 > 0$ such that, whenever α is sufficiently small

$$\frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho^2}(\rho_\alpha, \bar{s}_\alpha)(\zeta, \zeta) + \frac{\partial^2 \mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha, \bar{s}_\alpha)(v, v) \geq \int_0^{+\infty} \left[(\zeta')^2 + \zeta^2 + |w'|^2 \right] d\tau - \alpha M \int_0^{+\infty} |w|^2 d\tau$$
$$\geq C_1 \|(\zeta, w)\|_{\mathcal{H}}^2$$

independently on the support of the function w. Otherwise, when we do not impose any assumption on the energy h_α, since, for every α, the function ρ_α tends to 0 decreasing, we can find τ_α sufficiently large, such that

$$\frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho^2}(\rho_\alpha, \bar{s}_\alpha)(\zeta, \zeta) + \frac{\partial^2 \mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha, \bar{s}_\alpha)(v, v) \geq C_2 \|(\zeta, w)\|_{\mathcal{H}}^2 \tag{58}$$

for some positive constant C_2, whenever the support of w is contained in $(\tau_\alpha, +\infty)$. $\qquad\square$

The asymptotic behavior of a collision solution, recalled in Proposition 3.7, suggests an extension of Theorem 6.4 to suitable families of collision motions. Also in this case a uniform condition on the asymptotic sets of central configuration will be assumed. With this aim we give the following definition.

Definition 6.5. *We say that the set of central configurations $K_\alpha \subset \mathcal{C}_\alpha$ has the property of the asymptotic minimality if for every x_α solution $(1)_\alpha$ asymptotic to the set K_α there exists $t_\alpha = t(x_\alpha)$ such that (53) holds for every φ with compact support in $(t_\alpha, 1)$.*

The next result is a generalization of Theorem 6.4 to a larger class of total collision motions.

Theorem 6.6. *Let $(K_\alpha)_{\alpha \in A}$, $A \subset (0, 2)$, be a family of sets of central configurations. If there exists $\delta > 0$ such that (51) holds then there exists $\bar{\alpha} \in (0, 2)$ such that K_α has the property of the asymptotic minimality for every $\alpha \in (0, \bar{\alpha}) \cap A$.*

Proof. Let x_α, for some $\alpha \in A$ be a solution of $(1)_\alpha$ asymptotic to the set K_α. We argue as in the proof of Theorem 6.4 and in this setting, replacing $w = \rho v$ and using (29), we integrate by parts and obtain

$$\frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho^2}(\rho_\alpha, s_\alpha)(\zeta, \zeta) + \frac{\partial^2 \mathcal{A}_\alpha}{\partial s^2}(\rho_\alpha, s_\alpha)(v, v)$$

$$= \int_0^{+\infty} \left\{ \left(\frac{4}{2-\alpha} \right)^2 (\zeta')^2 + \zeta^2 \left[|s_\alpha'|^2 + 2U_\alpha(s_\alpha) \right] + \rho_\alpha^2 \left[|v'|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(s_\alpha)(v, v) \right] \right\} d\tau$$

$$= \int_0^{+\infty} \left\{ \left(\frac{4}{2-\alpha} \right)^2 (\zeta')^2 + \zeta^2 \left[|s_\alpha'|^2 + 2U_\alpha(s_\alpha) \right] + |w'|^2 + \frac{\rho_\alpha''}{\rho_\alpha} |w|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(s_\alpha)(w, w) \right\} d\tau$$

$$= \int_0^{+\infty} \left\{ \left(\frac{4}{2-\alpha} \right)^2 (\zeta')^2 + |w'|^2 + \left[\zeta^2 + \left(\frac{2-\alpha}{4} \right)^2 |w|^2 \right] \left[|s_\alpha'|^2 + 2U_\alpha(s_\alpha) \right] \right\} d\tau$$

$$+ \int_0^{+\infty} \left\{ \left(\frac{2-\alpha}{4} \right)^2 \beta h_\alpha \rho_\alpha^{\beta-2} |w|^2 + \nabla^2 U_{\alpha|\mathcal{E}}(s_\alpha)(w, w) \right\} d\tau.$$

Since $|s_\alpha'|^2 + 2U_\alpha(s_\alpha) > 0$, the uniform assumption (51) and Lemma 6.2 still imply the existence of a positive constant C_2 such that inequality (58) holds. On the other hand, integrating by parts

the mixed term $\frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho \partial s}(\rho, s)(\zeta, v)$ and recalling that v has compact support we obtain

$$
\begin{aligned}
\frac{1}{2} \frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho \partial s}(\rho_\alpha, s_\alpha)(\zeta, v) &= \int_0^{+\infty} \rho_\alpha \zeta \left(s_\alpha' \cdot v' + \nabla U_{\alpha|\mathcal{E}}(s_\alpha) \cdot v \right) d\tau \\
&= \int_0^{+\infty} \zeta s_\alpha' \cdot \left(w' - \frac{\rho_\alpha'}{\rho_\alpha} w \right) + \zeta \nabla U_{\alpha|\mathcal{E}}(s_\alpha) \cdot w \, d\tau \\
&= \int_0^{+\infty} \left(-\zeta' s_\alpha' - \zeta s_\alpha'' - \zeta \frac{\rho_\alpha'}{\rho_\alpha} s_\alpha' + \zeta \nabla U_{\alpha|\mathcal{E}}(s_\alpha) \right) \cdot w \, d\tau.
\end{aligned}
\tag{59}
$$

Replacing the second Euler–Lagrange equation (50) divided by ρ^2, that is

$$
-2 \frac{\rho_\alpha'}{\rho_\alpha} s_\alpha' - s_\alpha'' + \nabla U_{\alpha|\mathcal{E}}(s_\alpha) = |s_\alpha'|^2 s_\alpha,
$$

into (59) we obtain

$$
\frac{1}{2} \frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho \partial s}(\rho_\alpha, s_\alpha)(\zeta, v) = \int_0^{+\infty} \left(-\zeta' s_\alpha' + \zeta \frac{\rho_\alpha'}{\rho_\alpha} s_\alpha' + \zeta |s_\alpha'|^2 s_\alpha \right) \cdot w \, d\tau.
$$

The Hölder inequality gives immediately

$$
\begin{aligned}
\left| \int_0^{+\infty} \zeta' s_\alpha' \cdot w \, d\tau \right| &\leq \left(\int_0^{+\infty} |s_\alpha'|^2 \, d\tau \right)^{1/2} \left(\int_0^{+\infty} (\zeta')^2 |w|^2 \, d\tau \right)^{1/2} \\
&\leq \|w\|_\infty \left(\int_0^{+\infty} |s_\alpha'|^2 \, d\tau \right)^{1/2} \left(\int_0^{+\infty} (\zeta')^2 \, d\tau \right)^{1/2} \\
\left| \int_0^{+\infty} \zeta \frac{\rho_\alpha'}{\rho_\alpha} s_\alpha' \cdot w \, d\tau \right| &\leq \left(\int_0^{+\infty} |s_\alpha'|^2 \, d\tau \right)^{1/2} \left(\int_0^{+\infty} (\zeta)^2 \left(\frac{\rho_\alpha'}{\rho_\alpha} \right)^2 |w|^2 \, d\tau \right)^{1/2} \\
&\leq \|w\|_\infty \left(\int_0^{+\infty} |s_\alpha'|^2 \, d\tau \right)^{1/2} \left(\int_0^{+\infty} (\zeta)^2 \left(\frac{\rho_\alpha'}{\rho_\alpha} \right)^2 \, d\tau \right)^{1/2} \\
\left| \int_0^{+\infty} \zeta |s_\alpha'|^2 s_\alpha \cdot w \, d\tau \right| &\leq \|\zeta\|_\infty \|w\|_\infty \int_0^{+\infty} |s_\alpha'|^2 \, d\tau.
\end{aligned}
$$

and then

$$
\frac{1}{2} \left| \frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho \partial s}(\rho_\alpha, s_\alpha)(\zeta, v) \right| \leq
$$

$$
\|w\|_\infty \left[\left(\int_0^{+\infty} (\zeta')^2 \, d\tau \right)^{\frac{1}{2}} + \left(\int_0^{+\infty} \zeta^2 \left(\frac{\rho_\alpha'}{\rho_\alpha} \right)^2 \, d\tau \right)^{\frac{1}{2}} \right] \left(\int_0^{+\infty} |s_\alpha'|^2 \, d\tau \right)^{\frac{1}{2}} + \|\zeta\|_\infty \int_0^{+\infty} |s_\alpha'|^2 \, d\tau.
$$

Recall that α is fixed (and so small that (58) holds). Pick now $\varepsilon > 0$. Since ρ_α'/ρ_α converges to a finite limit as $\tau \to +\infty$, and $\int_0^{+\infty} |s_\alpha'|^2 < +\infty$ (Proposition 3.7), there exists τ_α depending on x_α such that

$$
\int_0^{+\infty} |s_\alpha'|^2 \, d\tau = \int_{\operatorname{supp} w} |s_\alpha'|^2 \, d\tau < \varepsilon
$$

whenever $\operatorname{supp} w \subset (\tau_\alpha, +\infty)$. Hence, for all such w's,

$$
\left| \frac{\partial^2 \mathcal{A}_\alpha}{\partial \rho \partial s}(\rho_\alpha, s_\alpha)(\zeta, v) \right| \leq C_3 \sqrt{\varepsilon} \, \|w\|_\infty \|\zeta\|_{H^1} \leq C_4 \sqrt{\varepsilon} \, \|(\zeta, w)\|_{\mathcal{H}}^2
\tag{60}
$$

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for some positive constants C_3, C_4. From (58) and (60) we obtain

$$d^2\mathcal{A}_\alpha(x_\alpha)((\zeta, v), (\zeta, v)) \geq \left(C_2 - 2C_4\sqrt{\varepsilon}\right)\|(\zeta, w)\|_{\mathcal{H}}^2 > 0.$$

$\square$

In Theorem 6.6, we cannot exclude that, as $\alpha \to 0$, the support of the variation φ moves off to the collision time $t = 1$. It is natural to investigate under what circumstances it is possible to single out a time t^*, independent of α, such that the second differential $d^2\mathcal{A}_\alpha(x_\alpha)(\varphi, \varphi) > 0$ is positive for any variation φ supported in $(t^*, 1)$. It will turn out that the following uniform condition on the behavior of ρ_α plays a crucial rôle.

(UC) As $\tau \to +\infty$, $\rho_\alpha(\tau) \to 0$ uniformly with respect to $\alpha \in (0, 2)$. More precisely, for all $\sigma > 0$ there exists $\tau_\sigma > 0$ such that for all $\alpha \in (0, 1)$ and all $\tau \geq \tau_\sigma$ there results $\rho_\alpha(\tau) < \sigma$.

Since it is clear that it would be useless to take the limit as $\alpha \to 0$ inside (1), we need to single out a non-trivial *limiting problem* that describes the asymptotic properties of one-collision solutions. Therefore we introduce the scaled potential

$$\widetilde{U}_\alpha(x) = \frac{1}{\alpha}\sum_{i<j}\frac{m_i m_j}{|x_i - x_j|^\alpha} = \frac{1}{\alpha}U_\alpha(x). \tag{61}$$

The corresponding action reads

$$\widetilde{\mathcal{A}}_\alpha(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt + \frac{1}{\alpha}\int_0^1 U_\alpha(x)\, dt \tag{62}$$

When we replace $\widetilde{U}_\alpha$ to U_α in $(1)_\alpha$, the solutions of the new dynamical system are strictly linked to the solutions of the old one as the next Lemma asserts. Its very simple proof is omitted.

Lemma 6.7. *If $\tilde{x} = \alpha^{-\frac{1}{\alpha+2}}x$, then $\widetilde{\mathcal{A}}_\alpha(\tilde{x}) = \alpha^{-\frac{2}{\alpha+2}}\mathcal{A}_\alpha(x)$. In particular, if x_α is a solution of $(1)_\alpha$, then $\tilde{x}_\alpha = \alpha^{-\frac{1}{\alpha+2}}x_\alpha$ solves*

$$\ddot{\tilde{x}}_\alpha = \nabla\widetilde{U}_\alpha(\tilde{x}_\alpha). \tag{63}$$

Remark 6.8. *It is evident that a solution of $\ddot{x} = \nabla\widetilde{U}_\alpha(x)$ is also a solution of $\ddot{x} = \nabla\widehat{U}_\alpha(x)$, where*

$$\widehat{U}_\alpha(x) = \frac{1}{\alpha}\sum_{i<j}m_i m_j\left(\frac{1}{|x_i - x_j|^\alpha} - 1\right). \tag{64}$$

For each $x = (x_1, \ldots, x_n) \in \mathbb{R}^{Nd}$, there results

$$\lim_{\alpha \to 0}\widehat{U}_\alpha(x) = -\sum_{i<j}m_i m_j \log|x_i - x_j| =: U_{\log}(x). \tag{65}$$

However, the potential $\widehat{U}_\alpha$ is lacking the homogeneity property which seems to be essential in the definition of the new variables (ρ, s), see Section 3.

We consider a family $(\tilde{x}_\alpha)_\alpha$, $\alpha \in (0, 2)$, such that, fixed α, $\tilde{x}_\alpha$ solves

$$\begin{cases} \ddot{\tilde{x}}_\alpha = \nabla\widetilde{U}_\alpha(\tilde{x}_\alpha) \\ \tilde{x}_\alpha(0) = x_\alpha^0 \in \mathbb{R}^{Nd}, \\ \dot{\tilde{x}}_\alpha(0) = v_\alpha^0 \in \mathbb{R}^{Nd}. \end{cases} \tag{66}$$

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We express $\tilde{x}_\alpha$ in terms of the generalized polar coordinates $(\tilde{\rho}_\alpha, \tilde{s}_\alpha)$ and the new time τ. Hence $\tilde{\rho}_\alpha$ and $\tilde{s}_\alpha$ satisfy the Euler–Lagrange equations (49) and (50) with $\widetilde{U}_\alpha$ instead of U_α. We make the following assumptions on the initial condition:

(IC1) $\tilde{\rho}_\alpha(0) = 1$ and $\tilde{\rho}'_\alpha(\tau) < 0$ for all α and for all $\tau \geq 0$.

(IC2) $\tilde{s}_\alpha(0) \to s_0$ and $\tilde{s}'_\alpha(0) \to v_0$ as $\alpha \to 0$.

Define
$$\Gamma_\alpha(\tau) = \frac{1}{2}|\tilde{s}'_\alpha(\tau)|^2 - \widehat{U}_\alpha(\tilde{s}_\alpha(\tau)) \tag{67}$$

where $\hat{U}_\alpha$ has been introduced in (64). In this setting we prove the next four Lemmas.

Lemma 6.9. *There exists a constant $C > 0$, independent of α, such that*
$$\int_0^{+\infty} -\frac{\tilde{\rho}'_\alpha}{\tilde{\rho}_\alpha}|\tilde{s}'_\alpha|^2 \, d\tau \leq C. \tag{68}$$

Proof. By differentiating Γ_α (with respect to τ) and making use of (50) with U_α replaced by $\tilde{U}_\alpha$, we compute
$$\frac{d\Gamma_\alpha}{d\tau} = -2\frac{\tilde{\rho}'_\alpha}{\tilde{\rho}_\alpha}|\tilde{s}'_\alpha|^2. \tag{69}$$

Therefore
$$\int_0^{+\infty} -\frac{\tilde{\rho}'_\alpha}{\tilde{\rho}_\alpha}|\tilde{s}'_\alpha|^2 \, d\tau = \lim_{\tau \to +\infty} \Gamma_\alpha(\tilde{s}_\alpha(\tau)) - \Gamma_\alpha(\tilde{s}_\alpha(0)) \tag{70}$$

We will complete the proof by showing that the right–hand side of (70) has a finite limit as $\alpha \to 0$. Since $\Gamma_\alpha(0) = \frac{1}{2}|\tilde{s}'_\alpha(0)|^2 - \hat{U}_\alpha(\tilde{s}_\alpha(0))$, by virtue of assumption **(IC2)**, we have that $\Gamma_\alpha(0)$ has a limit as $\alpha \to 0$.

As regards the behavior of $\Gamma_\alpha(+\infty) := \lim_{\tau \to +\infty} \Gamma_\alpha(\tilde{s}_\alpha(\tau))$, we deduce from the asymptotic estimates (see Proposition 3.7 (c)) that $\lim_{\tau \to +\infty} \widehat{U}_\alpha(\tilde{s}_\alpha(\tau)) = \hat{b}_\alpha$ exists and is finite. We choose $\bar{s}_\alpha \in \mathcal{E}$ such that $\hat{b}_\alpha = \widehat{U}_\alpha(\bar{s}_\alpha)$. Since $\mathcal{E}$ is a compact set, we may assume that $\bar{s}_{\alpha_k} \to \bar{s}_0$ for a suitable subsequence $\alpha_k \to 0$. Moreover, it is known that $\lim_{\tau \to +\infty} |\tilde{s}'_{\alpha_k}(\tau)| = 0$ (see Proposition 3.7 (f)). We conclude as before that $\Gamma_{\alpha_k}(+\infty)$ has a finite limit as $\alpha_k \to 0$. $\qquad\square$

Lemma 6.10. *For any $\varepsilon > 0$ there exist $\tau_\varepsilon > 0$ and $\alpha_\varepsilon \in (0, 2)$ such that*
$$\frac{2-\alpha}{4\alpha}\left(1 - \tilde{\rho}_\alpha(\tau)^{4\alpha/(2-\alpha)}\right) \geq \frac{1}{\varepsilon} \tag{71}$$

for all $\alpha \in (0, \alpha_\varepsilon)$ and $\tau \geq \tau_\varepsilon$.

Proof. To save notation, we set $\gamma = \gamma(\alpha) = 4\alpha/(2-\alpha)$. First of all, we remark that if $\rho_1, \rho_2 \in (0, 1)$ with $\rho_1 < \rho_2$, then
$$\frac{1 - \rho_1^\gamma}{\gamma} > \frac{1 - \rho_2^\gamma}{\gamma}.$$

We fix $\varepsilon > 0$ and choose $\eta = \eta_\varepsilon > 0$ such that $-\log \eta > 1/\varepsilon$. From assumption **(UC)**, we can fix $\tau_\varepsilon > 0$ such that $\tilde{\rho}_\alpha(\tau) < \eta$ whenever $\alpha \in (0, 2)$ and $\tau \geq \tau_\varepsilon$. Furthermore, since as $\alpha \to 0$ $(1 - \eta^\gamma)/\gamma \to -\log \eta$, we can fix $\alpha_\varepsilon \in (0, 2)$ such that
$$\frac{1 - \eta^\gamma}{\gamma} \geq -\log \eta - \varepsilon.$$

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Finally, if $\tau \geq \tau_\varepsilon$ and $\alpha < \alpha_\varepsilon$, we get

$$\frac{1 - \tilde{\rho}_\alpha(\tau)^\gamma}{\gamma} > \frac{1 - \eta^\gamma}{\gamma} \geq -\log \eta - \varepsilon \geq \frac{1}{\varepsilon} - \varepsilon. \tag{72}$$

Inequality (71) is of course equivalent to (72). $\qquad\square$

Before proceeding, we notice that, since $h_\alpha = \frac{1}{2}|\dot{x}_\alpha|^2 - U_\alpha(x_\alpha)$ and $\tilde{x}_\alpha = \alpha^{-\frac{1}{\alpha+2}} x_\alpha$, there results $\tilde{h}_\alpha = \frac{1}{2}|\dot{\tilde{x}}_\alpha|^2 - \frac{1}{\alpha}\tilde{U}_\alpha(\tilde{x}_\alpha) = \alpha^{-\frac{1}{\alpha+2}} h_\alpha$. Similarly, from Remark 6.8 we also get $\hat{h}_\alpha = \frac{1}{2}|\dot{\tilde{x}}_\alpha|^2 - \hat{U}_\alpha(\tilde{x}_\alpha) = \tilde{h}_\alpha + \frac{1}{\alpha}\sum_{i<j} m_i m_j$. We will assume that $\hat{h}_\alpha = 0$, which amounts to

(H) The energy of the solution x_α is $h_\alpha = \frac{\sum_{i<j} m_i m_j}{\alpha^{\frac{\alpha}{\alpha+2}}}$.

Lemma 6.11. *Assume condition **(H)**. Then there exists a constant C such that*

$$2\widetilde{U}_\alpha(\tilde{s}_\alpha(\tau)) + \beta\tilde{h}_\alpha\tilde{\rho}_\alpha(\tau)^{\beta-2} \geq C + \frac{1}{\varepsilon}$$

for all $\alpha \in (0, \alpha_\varepsilon)$ and $\tau \geq \tau_\varepsilon$.

Proof. We can write

$$\widetilde{U}_\alpha(\tilde{s}_\alpha) + \frac{\beta}{2}\tilde{h}_\alpha\rho_\alpha^{\beta-2} = \sum_{i<j} m_i m_j \left[\frac{1}{\alpha}\frac{1}{|\tilde{s}_{\alpha,i} - \tilde{s}_{\alpha,j}|^\alpha} - \frac{1}{\alpha}\frac{2+\alpha}{(2-\alpha)}\tilde{\rho}_\alpha^{4\alpha/(2-\alpha)} \right]$$

$$= \sum_{i<j} m_i m_j \left[\frac{1}{\alpha}\left(\frac{1}{|\tilde{s}_{\alpha,i} - \tilde{s}_{\alpha,j}|^\alpha} - 1\right) + \frac{1}{\alpha} - \frac{1}{\alpha}\frac{2+\alpha}{(2-\alpha)}\left(\tilde{\rho}_\alpha^{4\alpha/(2-\alpha)} - 1\right) - \frac{1}{\alpha}\frac{2+\alpha}{2-\alpha} \right]$$

$$= \sum_{i<j} m_i m_j \left[\frac{1}{\alpha}\left(\frac{1}{|\tilde{s}_{\alpha,i} - \tilde{s}_{\alpha,j}|^\alpha} - 1\right) + \frac{1}{\alpha}\left(1 - \frac{2+\alpha}{2-\alpha}\right) - 4\frac{2+\alpha}{(2-\alpha)^2}\frac{2-\alpha}{4\alpha}\left(\tilde{\rho}_\alpha^{4\alpha/(2-\alpha)} - 1\right) \right].$$

We now observe that since $\tilde{s}_\alpha \in \mathcal{E}$, then $|\tilde{s}_{\alpha,i} - \tilde{s}_{\alpha,j}| \leq 2$, for all α and $i \neq j$, and

$$\frac{1}{\alpha}\left(\frac{1}{|\tilde{s}_{\alpha,i} - \tilde{s}_{\alpha,j}|^\alpha} - 1\right) \geq \frac{1 - 2^\alpha}{\alpha 2^\alpha}$$

where the right hand side converges to $-\log 2$. The conclusion follows from Lemma 6.10 and easy algebraic inequalities. $\qquad\square$

Remark 6.12. *We notice that assumption **(H)** implies in particular $\lim_{\alpha\to 0} h_\alpha = \sum_{i<j} m_i m_j$. More generally, the same proof adapts to the case in which $\hat{h}_\alpha = C$, a constant independent of α. Indeed, the "old" energy would be $h_\alpha = \left(C\alpha^{\frac{2}{\alpha+2}} - \frac{1}{\alpha^{\frac{\alpha}{\alpha+2}}}\right)\sum_{i<j} m_i m_j$, and the first term tends to zero as $\alpha \to 0$.*

Lemma 6.13. *Assume condition **(H)**. Then, for every $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ and $\alpha_\varepsilon \in (0,2)$ such that*

$$\int_{\tau_\varepsilon}^\infty |\tilde{s}_\alpha'|^2 \, d\tau < \varepsilon.$$

for all $\alpha \in (0, \alpha_\varepsilon)$.

Proof. Set $\phi_\alpha(\tau) = -\tilde{\rho}_\alpha'(\tau)/\tilde{\rho}_\alpha(\tau)$. By direct computation

$$\phi_\alpha'(\tau) = -\frac{\tilde{\rho}_\alpha''}{\tilde{\rho}_\alpha} + \left(\frac{\tilde{\rho}_\alpha'}{\tilde{\rho}_\alpha}\right)^2 = -\frac{\tilde{\rho}_\alpha''}{\tilde{\rho}_\alpha} + \phi_\alpha(\tau)^2$$

hence using equation (49) and Lemma 6.11 we have

$$
\begin{aligned}
\phi_\alpha'(\tau) &= \left(\frac{2-\alpha}{4}\right)^2 \left[-|\tilde{s}_\alpha'|^2 - 2\widetilde{U}_\alpha(\tilde{s}_\alpha) - \beta h_\alpha \tilde{\rho}_\alpha^{\beta-2}\right] + \phi_\alpha(\tau)^2 \\
&\leq -\left(C + \frac{1}{\varepsilon}\right) + \phi_\alpha(\tau)^2.
\end{aligned}
$$

Since, by **(IC1)**, $\tilde{\rho}_\alpha' < 0$ we deduce that ϕ_α is positive. We claim that

$$\phi_\alpha(\tau)^2 \geq C + \frac{1}{\varepsilon} \qquad \text{for every } \tau > 0. \tag{73}$$

If (73) is false, then $\phi_\alpha(\tau_0) < \sqrt{C + 1/\varepsilon}$ for some $\tau_0 > 0$. Consider a solution ψ of the Cauchy problem

$$\begin{cases} \psi' = \psi^2 - (C + \frac{1}{\varepsilon}) \\ \psi(\tau_0) = \phi_\alpha(\tau_0). \end{cases}$$

A basic comparison theorem for ODEs implies that $\phi_\alpha(\tau) \leq \psi(\tau)$ for all $\tau \geq \tau_0$. But $\psi(\tau) \to -(C + 1/\varepsilon)$ as $\tau \to +\infty$, and therefore ϕ_α becomes negative for sufficiently large times. This is a contradiction that proves (73). □

Theorem 6.14. *Let $(x_\alpha)_{\alpha \in A}$, $A \subset (0,2)$, be a family of total one-collision solutions of $(1)_\alpha$, and let $(\tilde{x}_\alpha)_{\alpha \in A}$ be the corresponding solutions for the potential $\widetilde{U}_\alpha$. Retain assumptions **(UC)**, **(IC1-2)** and **(H)**. If there exists $\delta > 0$ for which (51) holds, then there exist t^* and a sequence $(\alpha_k)_k \subset A$ $\alpha_k \to 0$ such that*

$$d^2 \mathcal{A}_{\alpha_k}(x_{\alpha_k})(\varphi, \varphi) > 0 \tag{74}$$

for every variation φ with support in $(t^*, 1)$.

Proof. As already remarked, we can consider the action $\widetilde{\mathcal{A}}_\alpha$ instead of $\mathcal{A}_\alpha$. Furthermore, (55), (56) and (57) hold with U_α replaced by $\widetilde{U}_\alpha$. When we compute the variation of the action functional we follow the proof of Theorem 6.6 to obtain

$$
\begin{aligned}
&\frac{\partial^2 \widetilde{\mathcal{A}}_\alpha}{\partial \rho^2}(\tilde{\rho}_\alpha, \tilde{s}_\alpha)(\zeta, \zeta) + \frac{\partial^2 \widetilde{\mathcal{A}}_\alpha}{\partial \tilde{s}_\alpha^2}(\tilde{\rho}_\alpha, \tilde{s}_\alpha)(v, v) \\
&\qquad = \int_0^{+\infty} \left\{ \left(\frac{4}{2-\alpha}\right)^2 (\zeta')^2 + |w'|^2 + (\zeta)^2 \left[|\tilde{s}_\alpha'|^2 + 2\widetilde{U}_\alpha(\tilde{s}_\alpha)\right] \right\} d\tau \\
&\qquad + \int_0^{+\infty} \nabla^2(\widetilde{U}_\alpha)_{|\mathcal{E}}(\tilde{s}_\alpha)(w, w) \, d\tau + \int_0^{+\infty} \left(\frac{2-\alpha}{4}\right)^2 |w|^2 \left[|\tilde{s}_\alpha'|^2 + 2\widetilde{U}_\alpha(\tilde{s}_\alpha) + \beta h_\alpha \tilde{\rho}_\alpha^{\beta-2}\right] d\tau.
\end{aligned}
$$

The first integral is positive and tends to $+\infty$ as $\alpha \to 0$; the second one is bounded from below (indeed, following the same idea of Lemma 6.2, $\nabla^2(\widetilde{U}_\alpha)_{|\mathcal{E}}(s)(w, w) \geq -m|w|^2$). The third one can be handled with the help of Lemma 6.11, giving us the estimate

$$\int_0^{+\infty} \left(\frac{2-\alpha}{4}\right)^2 |w|^2 \left[|\tilde{s}_\alpha'|^2 + 2\widetilde{U}_\alpha(\tilde{s}_\alpha) + \beta \tilde{h}_\alpha \tilde{\rho}_\alpha^{\beta-2}\right] d\tau \geq \left(K + \frac{1}{\varepsilon}\right) \int_0^{+\infty} |w|^2 \, d\tau,$$

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for some $K > 0$. In conclusion, there exists a constant $C_1 > 0$ such that

$$\frac{\partial^2 \tilde{\mathcal{A}}_\alpha}{\partial \rho^2}(\rho_\alpha, \bar{s}_\alpha)(\zeta, \zeta) + \frac{\partial^2 \tilde{\mathcal{A}}_\alpha}{\partial s^2}(\rho_\alpha, \bar{s}_\alpha)(v, v) \geq C_2 \|(\zeta, w)\|_{\mathcal{H}}^2.$$

Concerning the mixed derivative, we argue as in the proof of 6.6 to obtain

$$\frac{1}{2}\frac{\partial^2 \widetilde{\mathcal{A}}_\alpha}{\partial \rho \partial s}(\tilde{\rho}_\alpha, \tilde{s}_\alpha)(\zeta, v) = \int_{\text{supp}\, v} \left(-\zeta' \tilde{s}_\alpha' + \zeta \frac{\tilde{\rho}_\alpha'}{\tilde{\rho}_\alpha} \tilde{s}_\alpha' + \zeta |\tilde{s}_\alpha'|^2 \tilde{s}_\alpha \right) \cdot w \, d\tau.$$

We use the Hölder inequality and get

$$\frac{1}{2}\left| \frac{\partial^2 \widetilde{\mathcal{A}}_\alpha}{\partial \rho \partial s}(\tilde{\rho}_\alpha, \tilde{s}_\alpha)(\zeta, v) \right| \leq \left(\int_{\text{supp}\, v} |\tilde{s}_\alpha'|^2 \, d\tau \right)^{\frac{1}{2}} \times$$

$$\times \left[\|w\|_\infty \left(\int_{\text{supp}\, v} (\zeta')^2 \, d\tau \right)^{\frac{1}{2}} + \|v\|_\infty \|\zeta\|_\infty \left(\int_{\text{supp}\, v} (\tilde{\rho}_\alpha')^2 \, d\tau \right)^{\frac{1}{2}} \right.$$

$$\left. + \|w\|_\infty \|\zeta\|_\infty \left(\int_{\text{supp}\, v} |\tilde{s}_\alpha'|^2 \, d\tau \right)^{\frac{1}{2}} \right]. \quad (75)$$

We have seen in Lemma 6.13 that, for every $\varepsilon > 0$, there exists τ_ε, independent of α, such that

$$\int_{\text{supp}\, v} |\tilde{s}_\alpha'|^2 \, d\tau < \varepsilon \qquad \text{for all } \alpha,$$

provided that $\text{supp}\, v \subset (\tau_\varepsilon, +\infty)$. To conclude, we notice that we are integrating over the compact set $\text{supp}\, v$ which is disjoint from the collision time. Basic results in the theory of ODEs (see Theorem 8.4 in [1]) imply that x_α' converges locally uniformly — and locally in L^2 — to some limit as $\alpha \to 0$. Thus, also ρ_α' converges in $L^2(\text{supp}\, v)$ to a limit, and in particular

$$\sup_\alpha \int_{\text{supp}\, v} (\tilde{\rho}_\alpha')^2 \, d\tau < +\infty.$$

This and (75) give that

$$\frac{1}{2}\left| \frac{\partial^2 \widetilde{\mathcal{A}}_\alpha}{\partial \rho \partial s}(\tilde{\rho}_\alpha, \tilde{s}_\alpha)(\zeta, v) \right| \leq C\sqrt{\varepsilon}\, \|(\zeta, v)\|_{\mathcal{H}}^2.$$

$\square$

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